Root-Locus Design

The root-locus can be used to determine the value of the loop gain K, which results in a satisfactory closed-loop behavior. This is called the *proportional compensator* or *proportional controller* and provides gradual response to deviations from the set point. There are practical limits as to how large the gain can be made. In fact, very high gains lead to instabilities. If the root-locus plot is such that the desired performance cannot be achieved by the adjustment of the gain, then it is necessary to reshape the root-loci by adding the additional controller $G_c(s)$ to the open-loop transfer function. $G_c(s)$ must be chosen so that the root-locus will pass through the proper region of the s-plane. In many cases, the speed of response and/or the damping of the uncompensated system must be increased in order to satisfy the specifications. This requires moving the dominant branches of the root locus to the left.

The proportional controller has no sense of time, and its action is determined by the present value of the error. An appropriate controller must make corrections based on the past and future values. This can be accomplished by combining proportional with integral action PI or proportional with derivative action PD. One of the most common controllers available commercially is the PID controller. Different processes are suited to different combinations of proportional, integral, and derivative control. The control engineer's task is to adjust the three gain factors to arrive at an acceptable degree of error reduction simultaneously with acceptable dynamic response. The compensator transfer function is

$$G_c(s) = K_P + \frac{K_I}{s} + K_D s \qquad (1)$$

For PD or PI controllers, the appropriate gain is set to zero.

Other compensators, are lead, lag, and lead-lag compensators. A first-order compensator having a single zero and pole in its transfer function is

$$G_c(s) = \frac{s + Z_0}{s + P_0} \qquad (2)$$

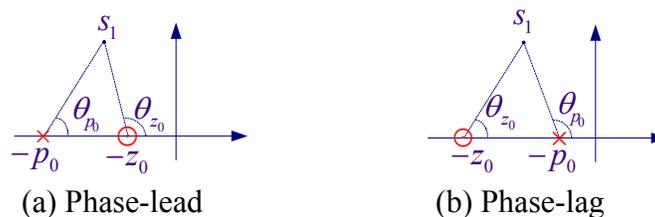The pole and zero are located in the left half s-plane as shown in Figure 1.

(a) Phase-lead   (b) Phase-lag
Figure 1 Compensator phase angle contribution

For a given $s_1 = \sigma_1 + j\omega_1$, the transfer function angle given by $\theta_c = (\theta_{z_0} - \theta_{p_0})$ is positive if $z_0 < p_0$ as shown in Figure 1 (a), and the compensator is known as the *phase-lead controller*. On the other hand if $z_0 > p_0$ as shown in Figure 1 (b), the compensator angle $\theta_c = (\theta_{z_0} - \theta_{p_0})$ is negative, and the compensator is known as the *phase-lag controller*

In general, the open-loop transfer function is given by

$$KG(s)H(s) = \frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

where m is the number of finite zeros and n is the number of finite poles of the loop transfer function. If $n > m$, there are $(n-m)$ zeros at infinity. The characteristic equation of the closed-loop transfer function is

$$1 + KG(s)H(s) = 0$$

Therefore

$$\frac{(s+p_1)(s+p_2)\cdots(s+p_n)}{(s+z_1)(s+z_2)\cdots(s+z_m)} = -K$$

From the above expression, it follows that for a point in the s-plane to be on the root-locus, when $0 < K < \infty$, it must satisfy the following two conditions.

$$K = \frac{|s+p_1||s+p_2|\cdots|s+p_n|}{|s+z_1||s+z_2|\cdots|s+z_m|} \quad \text{or}$$

$$K = \frac{\text{product of vector lengths from finite poles}}{\text{product of vector lengths from finite zeros}}$$

(3)

and

$$\sum \text{of zeros of } G(s)H(s) - \sum \text{angle of poles of } G(s)H(s) = r(180), \quad r = \pm1, \pm3, \cdots$$

or

$$\sum_{i=1}^{m} \theta_{zi} - \sum_{i=1}^{n} \theta_{pi} = 180r, \quad r = \pm1, \pm3, \cdots$$

(4)

The magnitude and angle criteria given by (3) and (4) are used in the graphical root-locus design.

In addition to the MATLAB control system toolbox **rlocus(num, den)** for root locus plot, MATALB control system toolbox contain the following functions which are useful for interactively finding the gain at certain pole locations and intersect with constant ω_n circles. These are:

sgrid generates a grid over an existing continuous s-plane root locus or pole-zero map. Lines of constant damping ratio ζ and natural frequency ω_n are drawn. sgrid('new') clears the current axes first and sets hold on.

sgrid(Z, Wn) plots constant damping and frequency lines for the damping ratios in the vector Z and the natural frequencies in the vector Wn.

[K, poles] = rlocfind(num, den) puts up a crosshair cursor in the graphics window which is used to select a pole location on an existing root locus. The root locus gain associated with this point is returned in **K** and all the system poles for this gain are returned in **poles**.

rltool or **sistool** opens the SISO Design Tool. This Graphical User Interface allows you to design single-input/single-output (SISO) compensators by interacting with the root locus, Bode, and Nichols plots of the open-loop system.

1. Gain Factor Compensation or P-Controller Design

The proportional controller is a pure gain controller. The design is accomplished by choosing a value K_0, which results in a satisfactory transient response. The specification may be either the step response damping ratio or the step response time constant or the steady-state error. The procedure for finding K_0 is as follows:

- Construct an accurate root-locus plot
- For a given ζ draw a line from origin at angle $\theta = \cos^{-1}\zeta$ measured from negative real axis.
- The desired closed-loop pole s_1 is at the intersection of this line and the root-locus.
- Estimate the vector lengths from s_1 to poles and zeros and apply the magnitude criterion as given by (3) to find K_0.

Example 1

The open-loop transfer function of a control system is given by

$$KGH(s) = \frac{K}{s(s+1)(s+4)}$$

(a) Obtain the gain K_0 of a proportional controller such that the damping ratio of the closed-loop poles will be equal 0.6. Obtain root-locus, step response and the time-domain specifications for the compensated system.

The root-locus plot is shown in Figure 2. For $\zeta = 0.6$,

$$\theta = \cos^{-1} 0.6 = 53.13°$$

The line drawn at this angle intersects the root-locus at approximately, $s_1 \simeq -0.41 + j0.56$. The vector lengths from s_1 to the poles are marked on the diagram
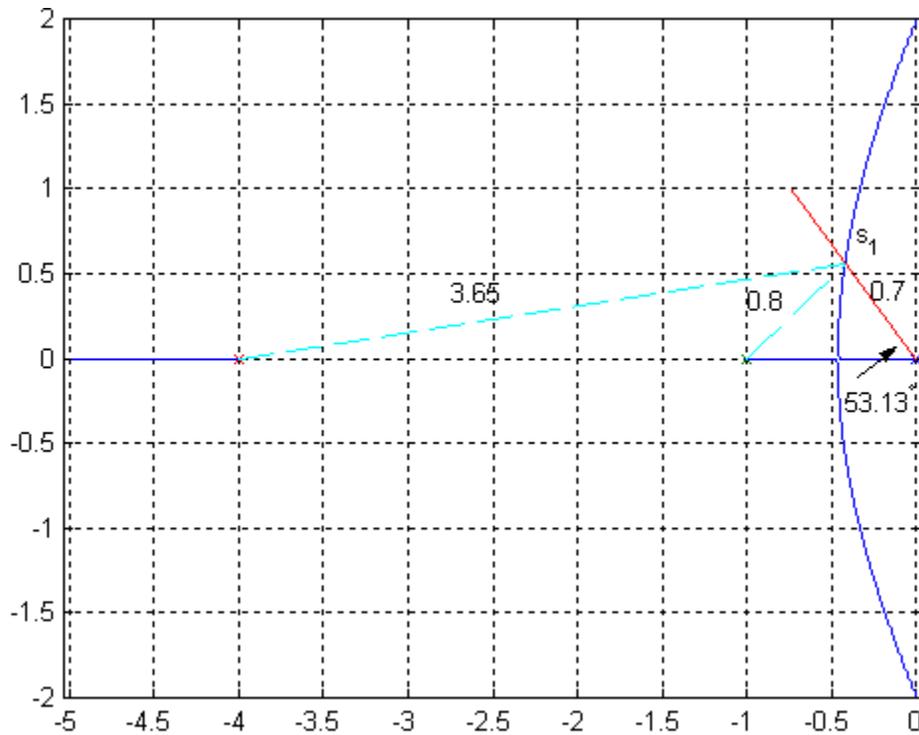
Figure 2 P-Controller Design

. From (1), we have

$$K = (0.7)(0.8)(3.65) = 2.04$$

This gain will result in the velocity error constant of $K_v = \dfrac{2.04}{4} = 0.51$. Thus, the steady-state error due to a ramp input is $e_{ss} = \dfrac{1}{K_v} = \dfrac{1}{0.51} = 1.96$.

The compensated closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{2.05}{s^3 + 5s^2 + 4s + 2.05}$$

(b) Use the MATLAB Control System Toolbox functions **rlocus** and **sgrid(zeta, wn)** to obtain the root-locus and the gain K_0 for $\zeta = 0.6$. Also use the **ltiview** function to obtain the system step response and the time-domain specifications.

The following commands
```
num=1;
den=[1 5 4 0];
rlocus(num, den);
hold on
sgrid(0.6, 1)    % plots constant line zeta=0.6 & constant line wn=1
```
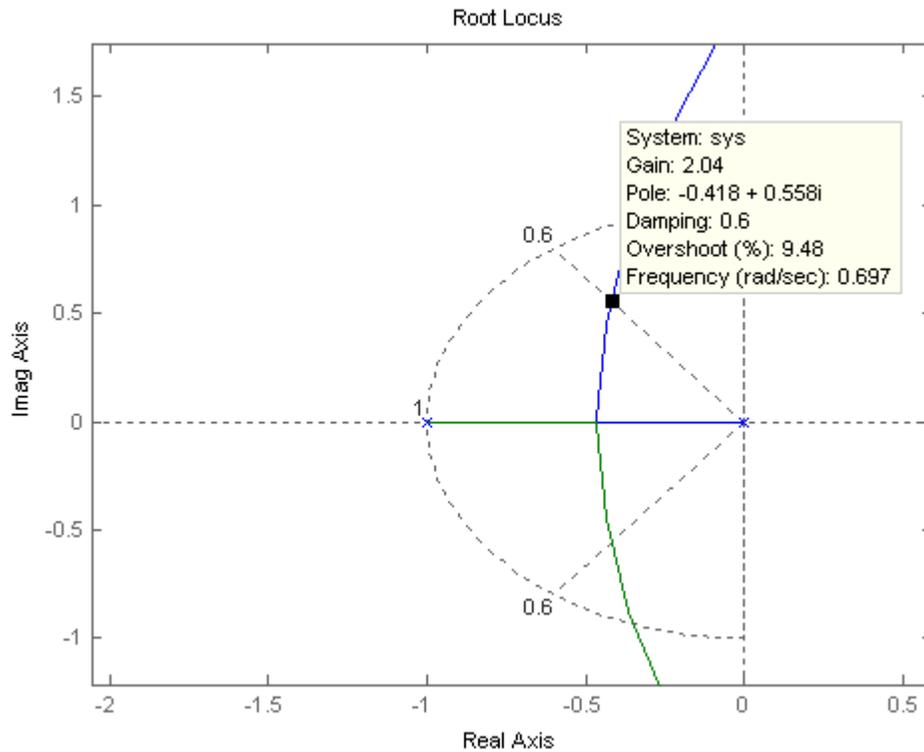
4

result in

Figure 3

Zoom in at the area of intersection, click at the intersection, hold and move the mouse at intersection and adjust for Damping: 0.6. The gain is found to be 2.04. In addition, the percentage overshoot and natural frequency are obtained, i.e., $PO = 9.48\%$ and $\omega_n = 0.697$.

To obtain the step response and time-domain specifications, we use the following commands.

```
numc=2.04;
denc=[1 5 4 2.04];
T=tf(numc, denc)
ltiview('step', T)
```

The result is shown in Figure 4. Right-click on the LTI Viewer, use Chracteristics to mark peak response, peak time, settling time, and rise time. From File Menu use Print to Figure to obtain a Figure plot.
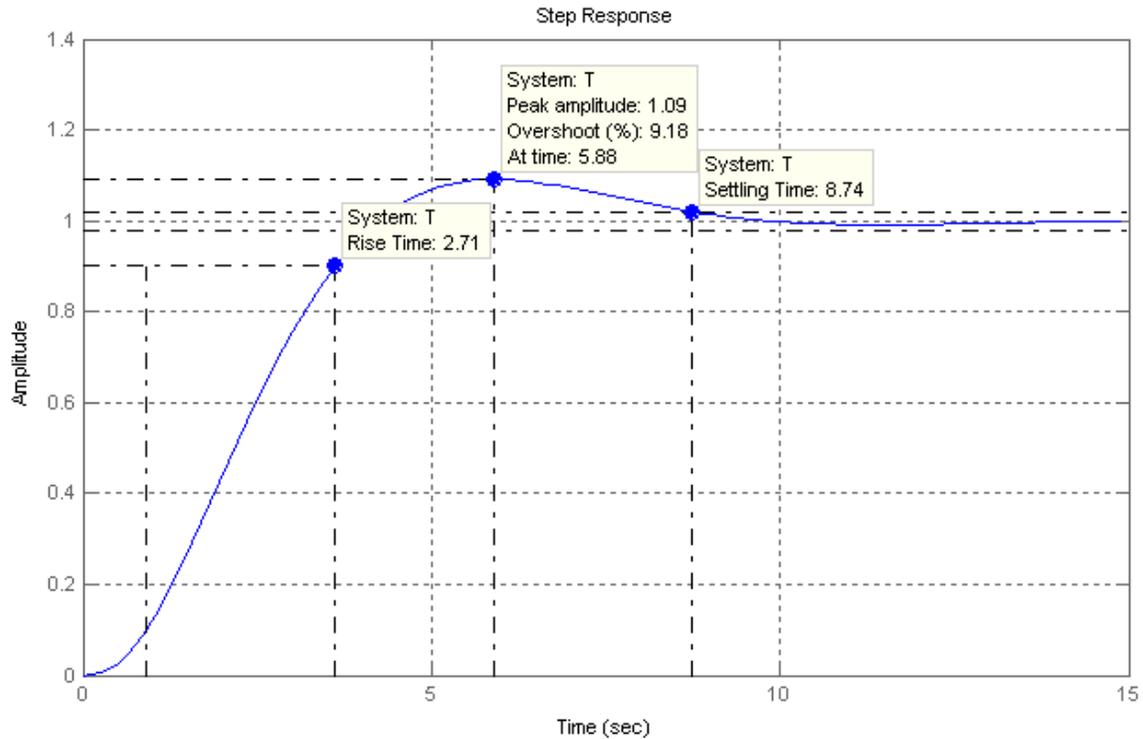
Figure4

2. *PD* Compensator Design

Here both the error and its derivative are used for control

$$G_c(s) = K_P + K_D s \qquad (5)$$

or

$$G_c(s) = K_D\left(s + \frac{K_P}{K_D}\right) = K_D(s + z_0)$$

where

$$Z_0 = \frac{K_P}{K_D} \qquad (6)$$

From above, it can be seen that the *PD* controller is equivalent to the addition of a simple zero at $Z_0 = K_P / K_D$ to the open-loop transfer function, which improves the transient response. From a different point of view, the PD controller may also be used to improve the steady-state error, because it anticipates large errors and attempts corrective action before they occur.

The procedure for the graphical root-locus *PD* compensator design is as follows:
- Construct an accurate root-locus plot
- From the design specifications; the desired damping ratio and time constant of the dominant closed-loop poles, obtain the desired location of the dominant closed-loop poles.

$$\zeta\omega_n = \frac{1}{\tau} \quad \text{and} \quad \theta = \cos^{-1}\zeta$$

$$b = \zeta\omega_n \tan\theta \quad \text{and} \quad s_1 = -\zeta\omega_n + jb$$
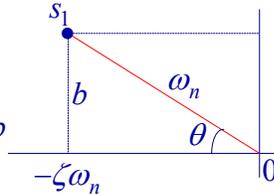
- Mark the poles and zeros of the open-loop plant transfer function. Find the location of the compensator zero Z_0 such that the angle criterion as given by (4) is satisfied.

$$\theta_{z0} + (\theta_{z1} + \theta_{z2} + \cdots) - (\theta_{p1} + \theta_{p2} + \cdots) = -180$$

- Estimate the vector lengths from s_1 to all poles and zeros and apply the magnitude criterion as given by (3) to find K_D. Find K_P from (6)

Example 2
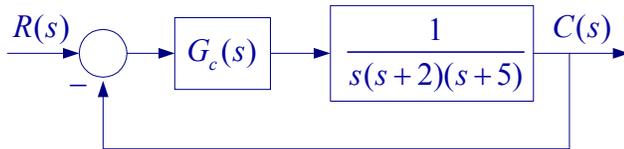Consider the control system shown in Figure 5.

Figure 5

(a) Assume the compensator is a simple proportional controller K, obtain all pertinent pints for root locus and draw the root-locus. Determine the location of the dominant poles to have critically damped response, and find the time constant corresponding to this location. Also determine the value of K and the corresponding time constant for dominant poles damping ratio of 0.707. Obtain the compensated system step response.

(b) $G_c(s)$ is a PD compensator. Design the compensator for the following time-domain specifications.

- Dominant poles damping ratio $\zeta = 0.707$
- Dominant poles time constant $\tau = 0.5$ second

(a) First we construct the root locus
- The root-loci on the real axis are to the left of an odd number of finite poles and zeros.
- $n - m = 3$, i.e., there are three zeros at infinity.
- Three asymptotes with angles $\theta = 180°$, and $\pm 60°$.
- The asymptotes intersect on the real axis at

$$\sigma_a = \frac{\sum \text{finite poles of } GH(s) - \sum \text{finite zeros of } GH(s)}{n-m} = \frac{-(2+5)}{3} - 2.33$$

- Breakaway point on the real axis is given by

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$$\frac{dK}{ds} = \frac{d}{ds}(s^3 + 7s^2 + 10s) = 0 \quad \Rightarrow \quad 3s^2 + 14s + 10 = 0$$

The roots of this equation are $s = -3.7863$, and $s = -0.8804$. But $s = -3.7863$ is not part of the root-locus for $K > 0$, therefore the breakaway point is at $s = -0.8804$. The Routh array gives the location of the $j\omega$-axis crossing.

$$\begin{array}{c|cc}
s^3 & 1 & 10 \\
s^2 & 7 & K \\
s^1 & 70-K & 0 \\
s^0 & K & 0
\end{array} \quad \Rightarrow \quad \text{for stability } 0 < K < 70 \text{ and } s = \pm j3.16$$

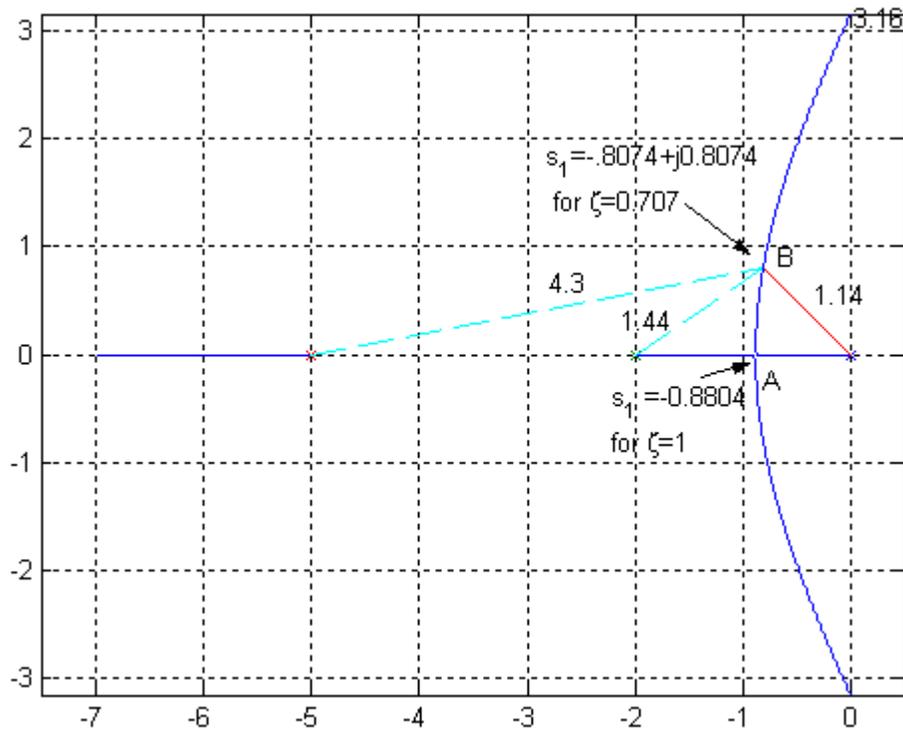The root-locus is shown in Figure 6.

Figure 6

For the dominant poles to have critically damped response, the dominant poles are at the breakaway position A, i.e., $s_1 = s_2 = -0.8804$. The time constant and the gain K are

$$\tau = \frac{1}{0.8804} = 1.136 \text{ second}$$

$$K = (0.8804)(1.1296)(4.1196) = 4.06$$

For dominant poles damping ratio of 0.707, s_1 is at position B. The time constant and the gain K are

$$\tau = \frac{1}{0.8074} = 1.24 \text{ second}$$

$$K = (1.14)(1.44)(4.3) = 7.06$$

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(b) The *PD* controller design

$$\zeta\omega_n = \frac{1}{\tau} = \frac{1}{0.5} = 2, \text{ and } \theta = coc^{-1}(0.707) = 45°$$

Therefore

$$s_1 = -2 + j2$$

The desired location of s_1 requires the root-locus to be shifted towards the left half s-plane, which requires the addition of zero by the *PD* controller as shown in Figure 7.
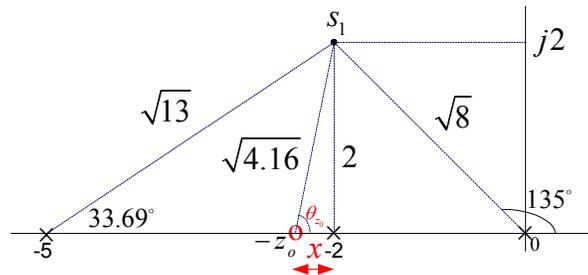

Figure 7

The position of z_0 is found by applying the angle criterion given by (4)

$$\theta_{z0} - (135 + 90 + 33.69) = -180 \quad \Rightarrow \quad \theta_{z0} = 78.69°$$

$$\tan 78.69 = \frac{2}{x} \quad \Rightarrow \quad x = 0.4, \text{ and } z_0 = 2.4 = \frac{K_P}{K_D}$$

The compensated open-loop transfer function is

$$G_c(s)GH(s) = \frac{K_D(s+2.4)}{s(s+2)(s+5)}$$

The vector lengths from s_1 are marked on the diagram as shown. Applying the magnitude criterion, we have

$$K_D = \frac{(\sqrt{8})(2)(\sqrt{13})}{\sqrt{4.16}} = 10$$

$$\frac{K_P}{K_D} = \frac{K_P}{10} = 2.4 \quad \Rightarrow \quad K_P = 24$$

Therefore, the controller transfer function is

$$G_c(s) = 24 + 10s$$

We use the following commands to obtain the closed-loop transfer function and the step response.

```
Gp = tf([0 0 1],[1  9  14])        % Plant transfer function
Gc = tf([10  24],[0  1])           % PD compensator
GpGc = series(Gp, Gc)              % Open-loop transfer function
T = feedback(GpGc, 1)              % closed-loop transfer function
ltiview('step', T)                 % obtains the step response
```
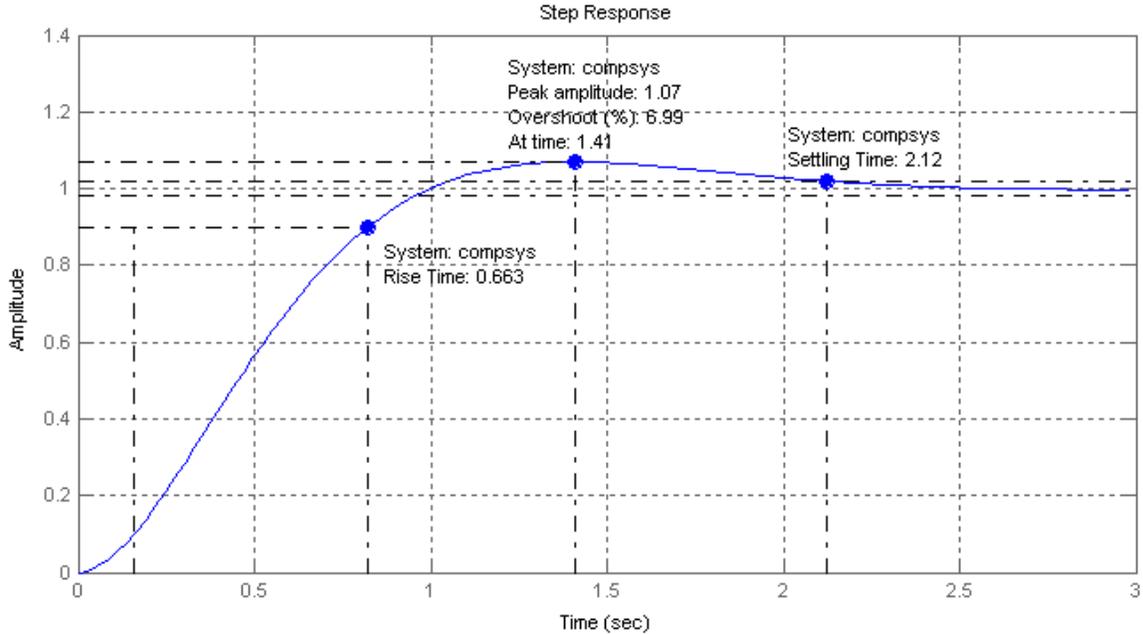
The result is shown in Figure 8.

Figure 8 Step response for the system of Example 2

3. PI Compensator Design

The integral of the error as well as the error itself is used for control, and the compensator transfer function is

$$G_c(s) = K_P + \frac{K_I}{s} \qquad (7)$$

or

$$G_c(s) = \frac{K_P s + K_I}{s} = \frac{K_P(s + \frac{K_I}{K_P})}{s} = \frac{K_P(s + z_0)}{s}$$

where

$$Z_0 = \frac{K_I}{K_P}$$

Integral control bases its corrective action on the cumulative error integrated over time. The controller increases the type of system by 1 and is used to eliminate the steady-state errors.

Example 3

For the control system shown in Figure 9 design a PI compensator for the following specifications:
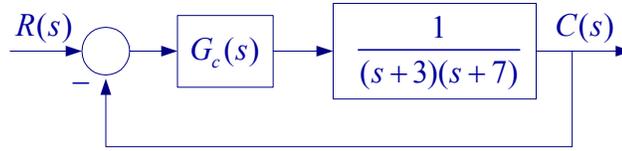
10

Figure 9

- Zero steady-state error due to a step input
- A pair of dominant closed-loop poles with a time constant of 0.25 seconds and a damping ratio of 0.8.

Obtain the compensated system step response.

$$\zeta\omega_n = \frac{1}{\tau} = \frac{1}{0.25} = 4 \text{, and } \theta = \tan^{-1}(0.8) = 36.87°$$

Therefore

$$s_1 = -4 + j4*\tan 36.87° \quad \Rightarrow \quad s_1 = -4 + j3$$

The poles of the open-loop transfer function and the controller pole at origin are marked in Figure 10.
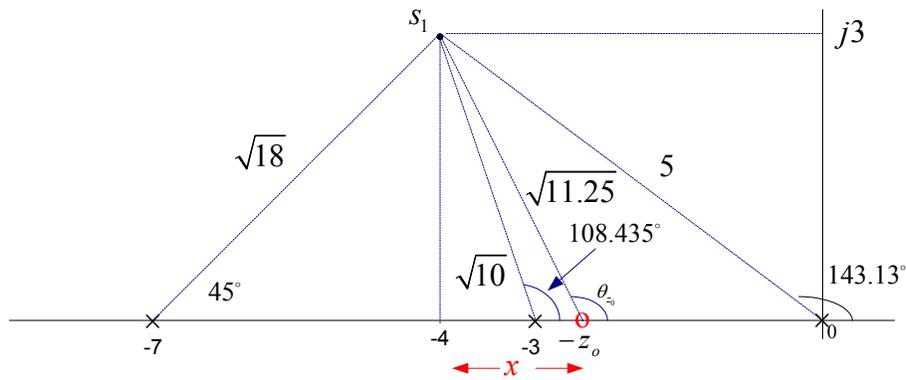

Figure 10

The position of controller zero for the desired location of s_1 is obtained by applying the angle criterion given by (4)

$$\theta_{z0} - (143.13 + 108.435 + 45) = -180 \quad \Rightarrow \quad \theta_{z0} = 116.565°$$

$$\tan(180 - 116.56) = \frac{3}{x} \quad \Rightarrow \quad x = 1.5, \text{ and } z_0 = 4 - 1.5 = 2.5$$

Therefore

$$\frac{K_I}{K_P} = 2.5$$

The compensated open-loop transfer function is

$$G_c(s)GH(s) = \frac{K_P(s + 2.5)}{s(s+3)(s+7)}$$

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The vector lengths from s_1 are marked on the diagram as shown. Applying the magnitude criterion, we have

$$K_P = \frac{(5)(\sqrt{10})(\sqrt{18})}{\sqrt{11.25}} = 20$$

$$\frac{K_I}{K_P} = \frac{K_I}{20} = 2.5 \quad \Rightarrow \quad K_I = 50$$

Therefore, the controller transfer function is

$$G_c(s) = 20 + \frac{50}{s}$$

The PI controller increases the system type from zero to 1. That is, we have a type 1 system and the steady-state error due to a step input is zero. We use the following commands to obtain the closed-loop transfer function and the step response.

```
Gp = tf([0 0 1],[1  10  21])      % Plant transfer function
Gc = tf([20  50],[1  0])          % PI compensator
GpGc = series(Gp, Gc)             % Open-loop transfer function
T = feedback(GpGc, 1)             % closed-loop transfer function
ltiview('step', T)                % obtains the step response
```

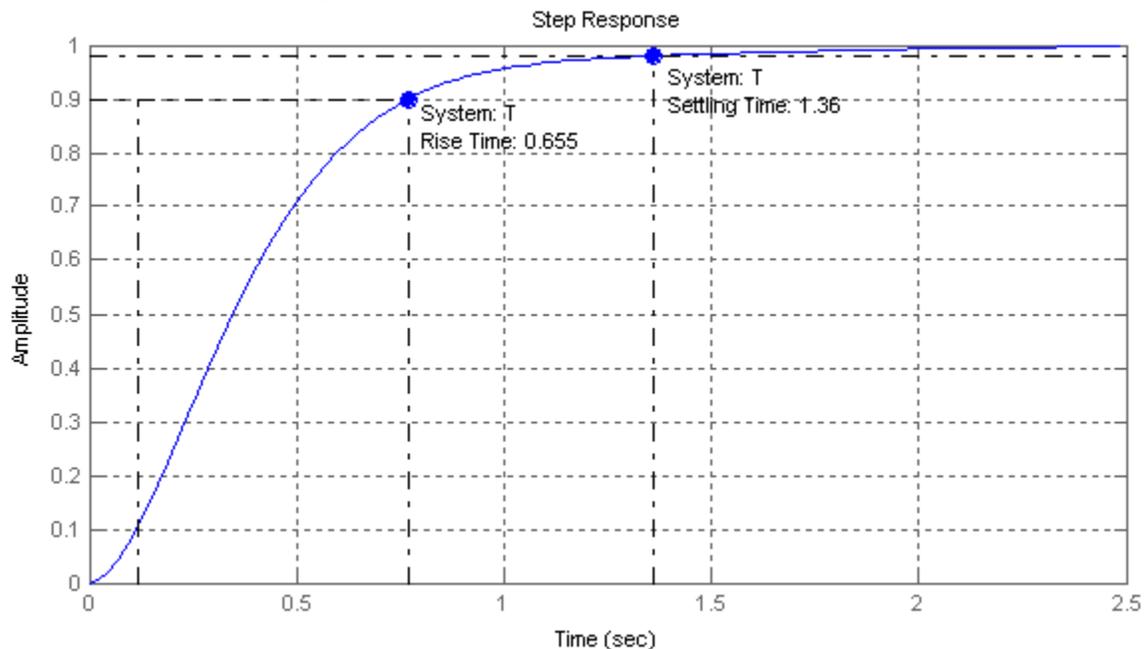The result is shown in Figure 11.

Figure 11 Step response for the system of Example 2

4. PID Compensator

The PID controller is used to improve the dynamic response as well as to reduce or eliminate the steady-state error. With a proportional controller increasing the controller gain will reduce the rise time and the steady-state error. However, in systems of third

order or higher, large gain will make the system unstable. Derivative action contributes phase-lead and will improve the transient response, reducing the overshoot and settling time. The integral action increases the system type by 1 and eliminates the steady-state error, but it may make the transient response worse. When you are designing a PID controller, first set K_P to a large value to produce a fast response without loosing stability. Then add derivative gain K_D and adjust its value to meet the transient response specifications. If required introduce the Integral gain K_I to eliminate the steady-state error. Repeat the design and fine-tune the gains to obtain the desired response.

5. Phase-Lead Design

In the phase-lead controller $z_0 > p_0$, thus the controller contributes a positive angle to the root-locus angle criterion and tends to shift the root-locus of the plant toward the left in the s-plane. Since $z_0 > p_0$, the compensator is a high-pass filter. The phase-lead compensator has the same purpose as the PD compensator. It is utilized to improve the transient response, to raise bandwidth and to increase the speed of response. A lead compensator approximates derivative control and reduces the high-frequency noise present in the PD compensator. The procedure or the graphical root-locus design is as follows:

- From the time-domain specifications obtain the desired location of the closed-loop dominant poles.
- Select the controller zero. Place the zero to the left of the smallest plant's pole (or on the pole for *pole-zero cancellation*)
- Locate the compensator pole so that the angle criterion (3) is satisfied.
- Determine the compensator gain K_c such that the magnitude criterion (4) is satisfied.
- If the overall response rise time, overshoot and settling time is not satisfactory, place the controller zero at a different location and repeat the design

Moving the controller zero to the left away from the origin in the s-plane results in a faster response with increase in overshoot. Moving the controller zero to the right towards the origin will result in a slow response and reduces or eliminate the overshoot. The compensator angle $\theta_{z0} - \theta_{p0}$ must be positive Therefore, there is a limit on how far to the left along the real axis the compensator zero may be moved and still be able to satisfy the angle criterion.

Example 4

For the control system of Example 2 design a phase lead compensator to meet the following time-domain specifications:

- Dominant poles damping ratio $\zeta = 0.707$
- Dominant poles time constant $\tau = 0.5$ second

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$$\zeta\omega_n = \frac{1}{\tau} = \frac{1}{0.5} = 2 \text{, and } \theta = coc^{-1}(0.707) = 45°$$

Therefore

$$s_1 = -2 + j2$$

The desired location of s_1 requires the root-locus to be shifted towards the left half s-plane, which requires the addition of phase lead controller as shown in Figure 12.
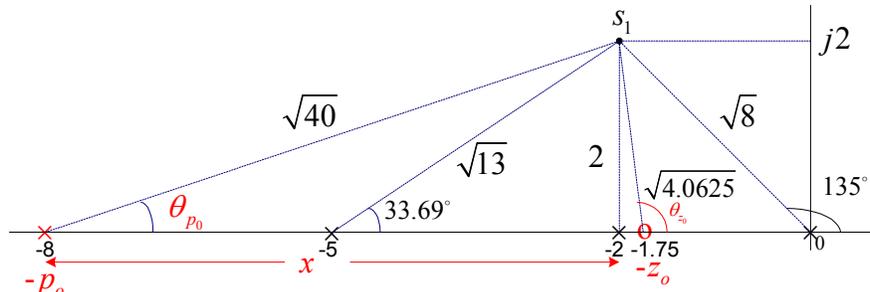

Figure 12

Let the controller zero be located at $z_0 = 1.75$. The position of the controller p_0 is found by applying the angle criterion given by (4)

$$\theta_0 = 180 - \tan^{-1}(2/0.25) = 97.125°$$

$$97.125 - (135 + 90 + 33.69 + \theta_{p0}) = -180 \quad \Rightarrow \quad \theta_{p0} = 18.435°$$

$$\tan 18.435 = \frac{2}{x} \quad \Rightarrow \quad x = 6 \text{, and } z_0 = 2 + 6 = 8$$

The compensated open-loop transfer function is

$$G_c(s)GH(s) = \frac{K_c(s+1.75)}{s(s+2)(s+5)(s+8)}$$

The vector lengths from s_1 are marked on the diagram as shown. Applying the magnitude criterion, we have

$$K_c = \frac{(\sqrt{8})(2)(\sqrt{13})(\sqrt{40})}{\sqrt{4.0625}} = 64$$

Therefore, the controller transfer function is

$$G_c(s) = \frac{64(s+1.75)}{(s+8)}$$

The controller dc gain is

$$a_0 = G_c(0) = \frac{(64)(1.75)}{8} = 14$$

We use the following commands to obtain the closed-loop transfer function and the step response.

```
Gp = tf([0 0  0 1],[1 7  10 0])          % Plant transfer function
Gc = tf(64*[1  1.75],[1  8])             % PI compensator
```

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```
GpGc = series(Gp, Gc)                    % Open-loop transfer function
T = feedback(GpGc, 1)                    % closed-loop transfer function
ltiview('step', T)                       % obtains the step response
```

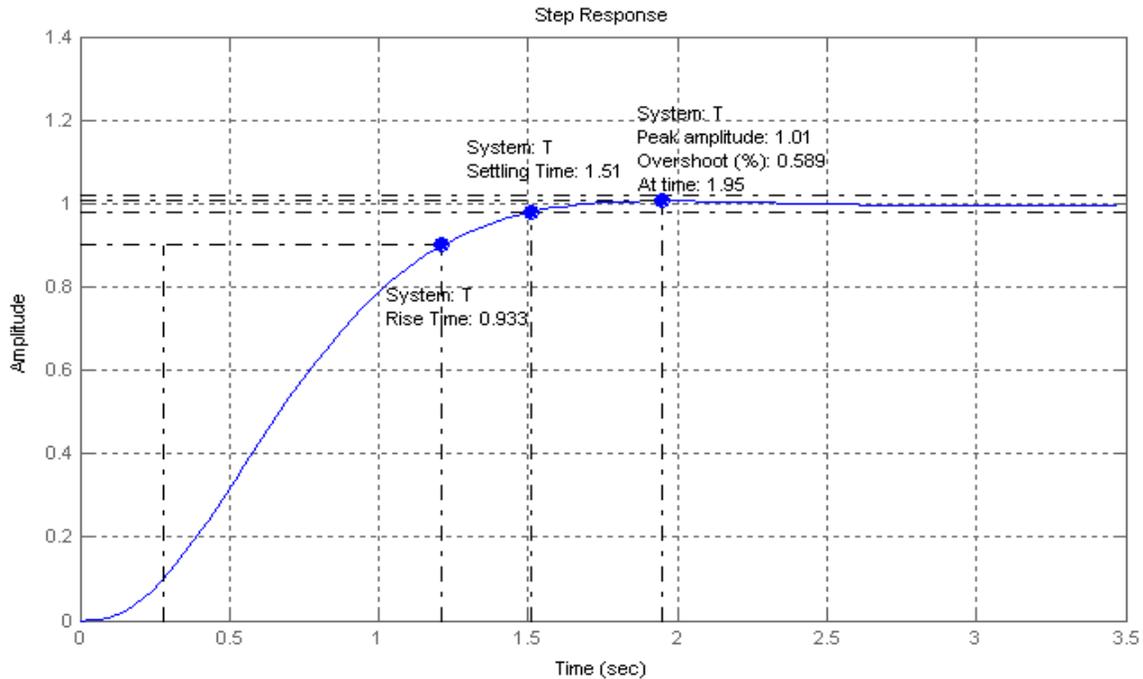The result is shown in Figure 13.

Figure 13 Step response for the system of Example 4.

6. Phase-lag compensator approximate design

The lag compensator is an approximate integral control. The phase-lag compensator is used when the system transient response is satisfactory but requires a reduction in the steady-state error. Since $p_0 < z_0$, the compensator is a low-pass filter. It adds a negative angle to the angle criterion and tends to shift the root-locus to the right in the s-plane.

In the phase-lag control, the controller poles and zeros are placed very close together, and the combination is located relatively close to the origin of the s-plane. Thus, the root-loci in the compensated system are shifted only slightly from their original locations. The compensator contributes a magnitude of

$$|G_c(s)| = \frac{K_c|s_1 + z_0|}{|s_1 + p_0|} \simeq K_c$$

The gain to satisfy the desired damping ratio is given by

$$K_0 \mid GH(s_1) \mid = 1 \quad \Rightarrow \quad \mid GH(s_1) \mid = \frac{1}{K_0}$$

For the compensated system, the magnitude criterion requires that

$$K \mid GH(s_1) \mid \| G_c(s_1) \mid = 1 \quad \Rightarrow \quad K \frac{1}{K_0} K_c = 1$$

or

$$K_c = \frac{K_0}{K} = \frac{\text{Gain to satisfy the desired damping ratio}}{\text{Gain to satisfy the desired steady-state error}} \qquad (7)$$

For a given desired location of a closed-loop pole s_1, the design can be accomplished by trial and error. The procedure for approximate phase-lag design is as follows:

- Obtain the root-locus and determine the gain K_0 to satisfy the desired damping ratio.

- Determine the gain K to satisfy the desired stead-state error.

- Evaluate the controller gain
$$K_c = \frac{\text{Gain to satisfy the desired damping ratio}}{\text{Gain to satisfy the desired steady-state error}} = \frac{K_0}{K}$$

- Select the controller zero z_0 close to origin.

- Based on the compensator DC gain of unity, $\frac{K_c z_0}{p_0} = 1$, find the controller pole
$$p_0 = K_c z_0$$

Example 5
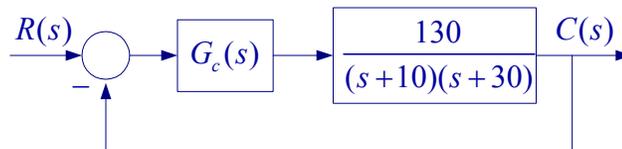Consider the control system shown in Figure 14.

Figure 14

(a) Assume the compensator is a simple proportional controller K, obtain all pertinent pints for root locus and draw the root-locus. Determine the gain K_0 for the step response damping ratio of 0.8. Obtain the steady-state error and the system step response.

- The root-loci on the real axis are to the left of an odd number of finite poles and zeros.

- $n - m = 2$, i.e., there are two zeros at infinity.

- Two asymptotes with angles $\theta = \pm 90°$.

- The asymptotes intersect on the real axis at
$$\sigma_a = \frac{\sum \text{finite poles of } GH(s) - \sum \text{finite zeros of } GH(s)}{n - m} = \frac{-(10 + 30)}{2} - 20$$

- Breakaway point on the real axis is given by
$$\frac{dK}{ds} = \frac{d}{ds}(s^2 + 40s + 300) = 0 \quad \Rightarrow \quad 2s + 40 = 0$$
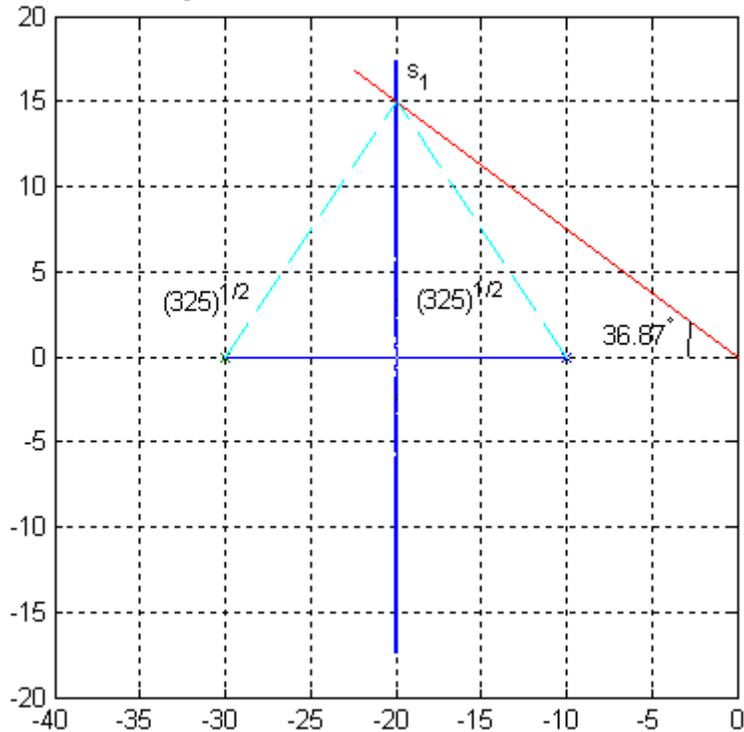Therefore the breakaway point is at $s = -20$.

The root-locus is shown in Figure 15

Figure 15

For $\zeta = 0.8 \quad \Rightarrow \quad \theta = \cos^{-1}(0.8) = 36.87°$

The intersection of the line drawn from origin at this angle with root locus gives the desired complex pole $s_1 = -20 + j15$. Applying the magnitude criterion (3), the gain K_0 is found

$$130 K_0 = \sqrt{325}\sqrt{325} \quad \Rightarrow \quad K_0 = 2.5$$

The position error constant is

$$K_p = \frac{(130)(2.5)}{(10)(30)} = 1.08333$$

The steady-state error is

$$e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+1.08333} = 0.48$$

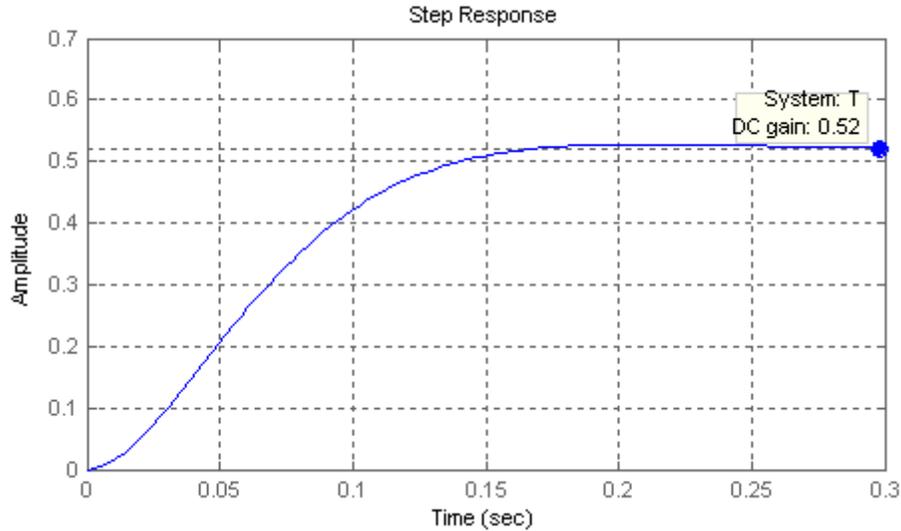The step response is shown in Figure 16.

Figure 16 The step response for Example 5 (a).

(b) It is required to have approximately the same dominant closed-loop pole locations and the same damping ratio ($\zeta = 0.8$) as in part (a). Design a phase-lag compensator such that the steady-state error due to a unit step input e_{ss} will be equal to 0.0845. Obtain the step response, and the time-domain specifications for the compensated system.

The gain K, which results in $e_{ss} = 0.0845$ is given by

$$e_{ss} = 0.0845 = \frac{1}{1+K_p} \quad \Rightarrow \quad K_p = 10.8343 = \frac{130K}{(10)(30)}$$

Thus the gain to realize the steady-state error specification is $K = 25$

Using the approximate method, the controller gain is given by

$$K_c = \frac{\text{Gain to satisfy the desired damping ratio}}{\text{Gain to satisfy the desired steady-state error}} = \frac{2.5}{25} = 0.1$$

Next choose a small value for the compensator zero, e.g., $z = 1.5$

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Based on the controller dc gain of unity $K_c z_0 / p_0 = 1$, the controller pole is found $p_0 = K_c z_0 = (0.1)(1.5) = 0.15$. Thus the controller transfer function is

$$G_c(s) = \frac{0.1(s+1.5)}{(s+0.15)}$$

and the compensated open-loop transfer function is

$$G_c(s)KG(s) = \frac{(0.1)(s+1.5)(130)(25)}{(s+0.15)(s+10)(s+30)} = \frac{325s+478.5}{s^3+40.15s^2+306s+45}$$

The compensated closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{325s+478.5}{s^3+40.15s^2+631s+532.5}$$

The compensated characteristic equation roots are $-19.63 \pm j14.5$, and -0.894. The compensated step response is shown in Figure 17. The complex poles are shifted slightly to the left from the specified value of $-20 \pm j15$.
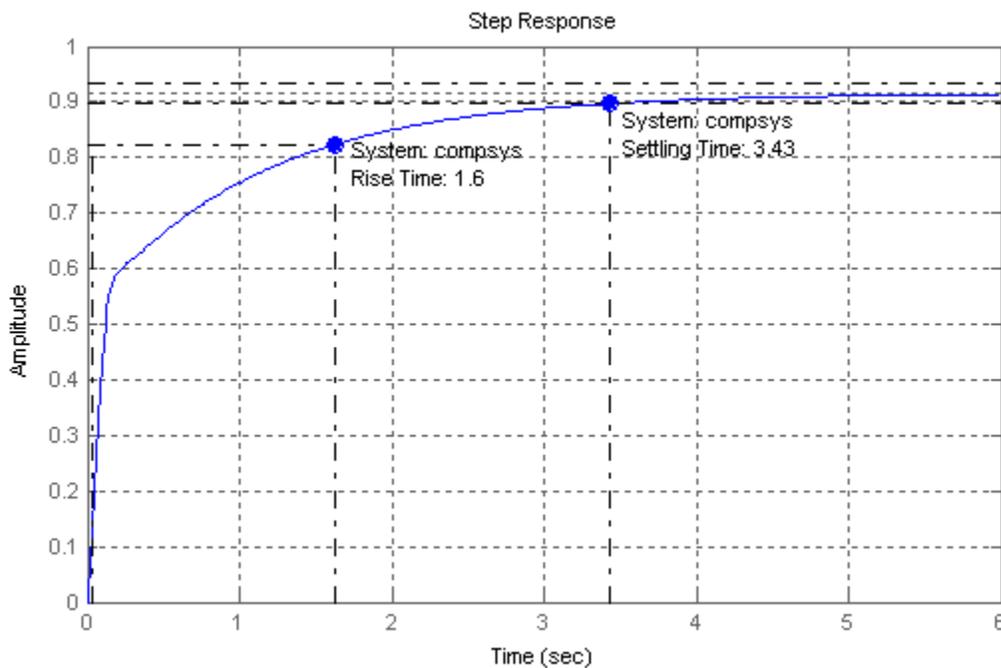

Figure 17 The compensated step response for Example 5 (b).

Note that that the complex poles are located approximately in the same location as in part (a). The steady-state error is greatly reduced, but because of the addition of the root at -0.894, the step response rise time and settling time are increased. If a faster response is desired, select the controller zero further to the left away from the origin. This would move the complex pole s_1 to the right further away from the specified value.

7. Phase-lead Compensator Analytical Design

The DC gain of the compensator $G_c(s) = \dfrac{K_c(s + z_0)}{(s + p_0)}$ is

$$a_0 = G_c(0) = \frac{K_c z_0}{p_0} \tag{8}$$

In the analytical design the controller dc gain a_0 is specified, usually in accordance to the steady-state error specification. Then, for a given location of the closed-loop pole

$$s_1 = |s_1| \angle \beta ,$$

z_0, and p_0 are obtained such that the equation

$$1 + G_c(s_1)GH(s_1) = 0$$

is satisfied. It can be shown that the above parameters are found from the following equations

$$z_0 = \frac{a_0}{a_1}, \qquad p_0 = \frac{1}{b_1}, \qquad \text{and} \qquad K_c = \frac{a_0 p_0}{z_0} \tag{9}$$

where

$$a_1 = \frac{\sin \beta + a_0 M \sin(\beta - \psi)}{s_1 M \sin \psi}$$

$$b_1 = -\frac{\sin(\beta + \psi) + a_0 M \sin \beta}{s_1 \sin \psi} \tag{10}$$

where M and ψ are the magnitude and phase angle of the open-loop plant transfer function evaluated at s_1, i.e.,

$$GH(s_1) = M \angle \psi \tag{11}$$

For the case that ψ is either $0°$ or $180°$, (10) is given by

$$a_1 s_1 \cos \beta \pm \frac{b_1 |s_1|}{M} \cos \beta \pm \frac{1}{M} + a_0 = 0 \tag{12}$$

where the plus sign applies for $\psi = 0°$ and the minus sign applies for $\psi = 180°$. For this case the zero of the compensator must also be assigned.

8. PDI Compensator Analytical Design

For a desired location of the closed-loop pole s_1, as given by (3), the following equations are obtained to satisfy

$$K_P = \frac{-\sin(\beta + \psi)}{M \sin \beta} - \frac{2K_I \cos \beta}{s_1}$$

$$K_D = \frac{\sin \psi}{|s_1| M \sin \beta} + \frac{K_I}{|s_1|^2}$$

(13)

For PD or PI controllers, the appropriate gain is set to zero. The above equations can be used only for the complex pole s_1. For the case that s_1 is real, the zero of the PD controller $(z_0 = K_P / KD)$ and the zero of the PI controller $(z_0 = K_I / K_P)$ are specified and the corresponding gains to satisfy angle and magnitude criteria are obtained accordingly. For the PID design, the value of K_I to achieve a desired steady state error is specified. Again, (13) is applied only for the complex pole s_1.

9. GUI program for root-locus compensator design (rldesigngui)

Based on the above equations, a Graphical User Interface program has been developed for the design of a first-order controller in the forward path of a closed-loop control system for proportional, phase-lag, phase-lead, PD, PI, and PID controllers. The GUI program named "rldesigngui", which has the following options, can invoke these programs:

Pushbutton P Controller – This option is used for the design of gain factor compensation. K_0 is obtained for the specified damping ratio ζ.

Pushbutton Phase Lag Controller – This option is used for the design of a phase-lag controller using the approximated method, $K_c = \frac{K_0}{K}$. $G_c(s)$ is designed for a desired damping ratio ζ and the gain K required for the steady-state error specification. The user must estimate the compensator zero. z_0 is selected far away from s_1 and close to origin.

Pushbutton Phase Lead Controller – This option is used for the design of a phase-lead controller for a desired location of the dominant complex closed loop poles. The DC gain of the controller $G_c(0)$ must be specified. $G_c(0) = \frac{K_c z_0}{p_0}$ is found from the steady-state error requirement.

Pushbutton PD Controller – This option is used for the design of a PD controller for a desired location of the dominant complex closed loop poles.

Pushbutton PI Controller – This option is used for the design of a PI controller for a desired location of the dominant complex closed loop poles.

Pushbutton PID Controller – This option is used for the design of a PID controller for a desired location of the dominant complex closed loop poles. The integral gain K_I must be specified.

For each case the open loop and the closed-loop compensated system transfer functions are displayed. Also, the variables Gc (controller transfer function), Tfo (compensated open-loop transfer function), and TFc (compensated closed-loop transfer function) are sent to the workspace. For each design the pushbutton System Responses can be used to obtain the time-domain and frequency-domain responses of the compensated system

Example 6

Use the **rldesigngui** to design a phase-lead controller for the system of Example 2 and the design specifications outlined in Example 4.

The open-loop transfer function of Example 2 is

$$GH(s) = \frac{1}{s(s+2)(s+5)}$$

The specification of $\zeta = 0.707$, and $\tau = 0.5$ for dominant closed-loop poles as specified in Example 4 resulted in the closed-loop pole location $s_1 = -2 + j2$. The analytical phase-lead controller design requires the specification of the controller dc gain. This is often obtained by specifying the steady-state error. We are going to use the controller dc gain obtained in the Example 4, i.e., $a_0 = 14$.

In MATLAB set the Current Directory to the folder where rldesigngui and the related files are located. At the MATLAB prompt type

>> rldesigngui

The following graphical window is displayed
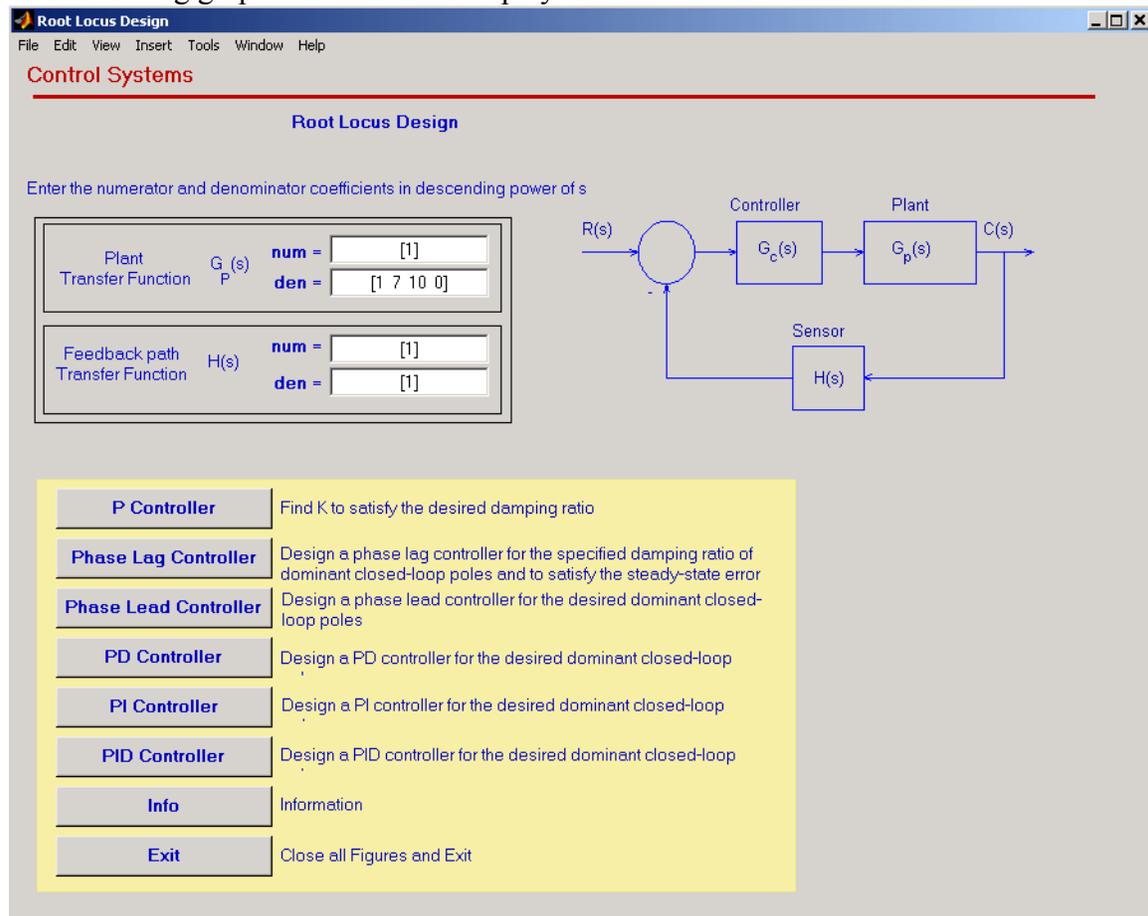
Enter the plant transfer function numerator and denominator coefficients. Select the Phase Lead Controller pushbutton. This opens the phase Lead Controller Design; enter the desired closed-loop pole and the controller dc gain.
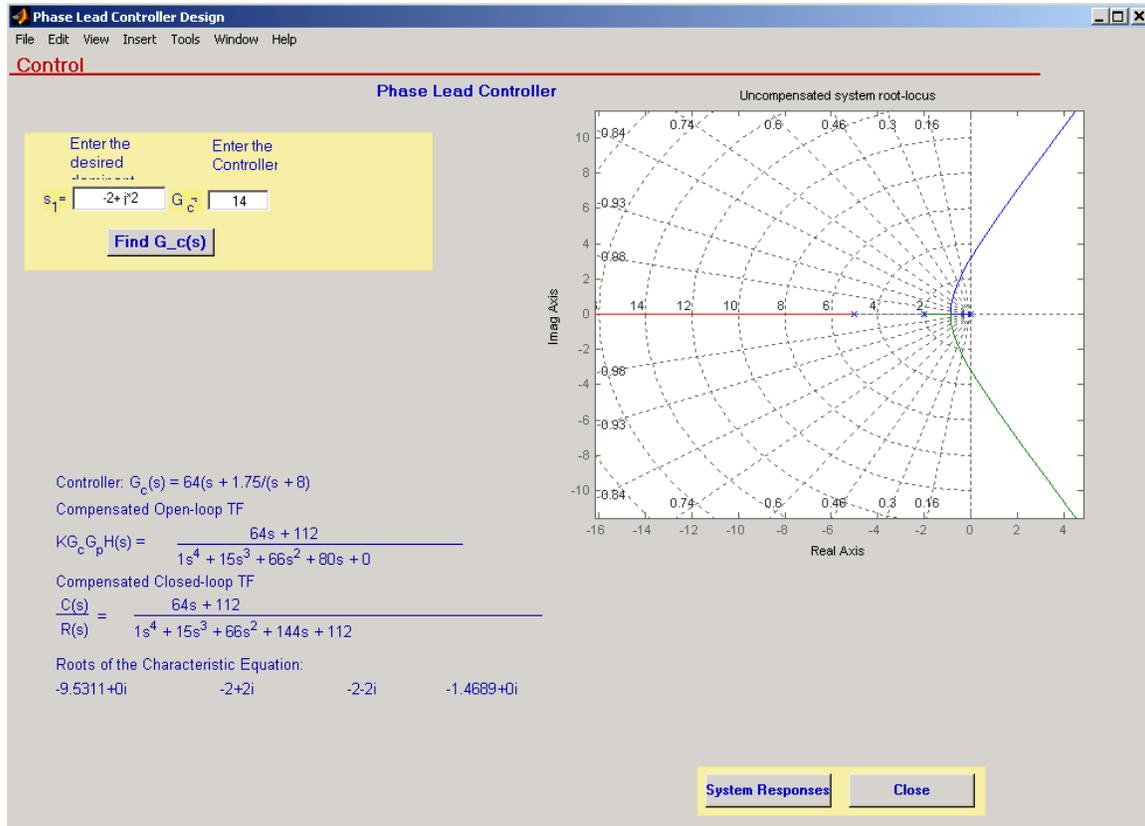
Pressing the Find $G_c(s)$ button, the controller transfer function, the compensated open loop and closed-loop transfer function, and the roots of the compensated characteristic equation are obtained as shown in the Figure. Pressing the System Responses pushbutton will activate the ltiviewer, which enables you to obtain all system response, and their characteristics.

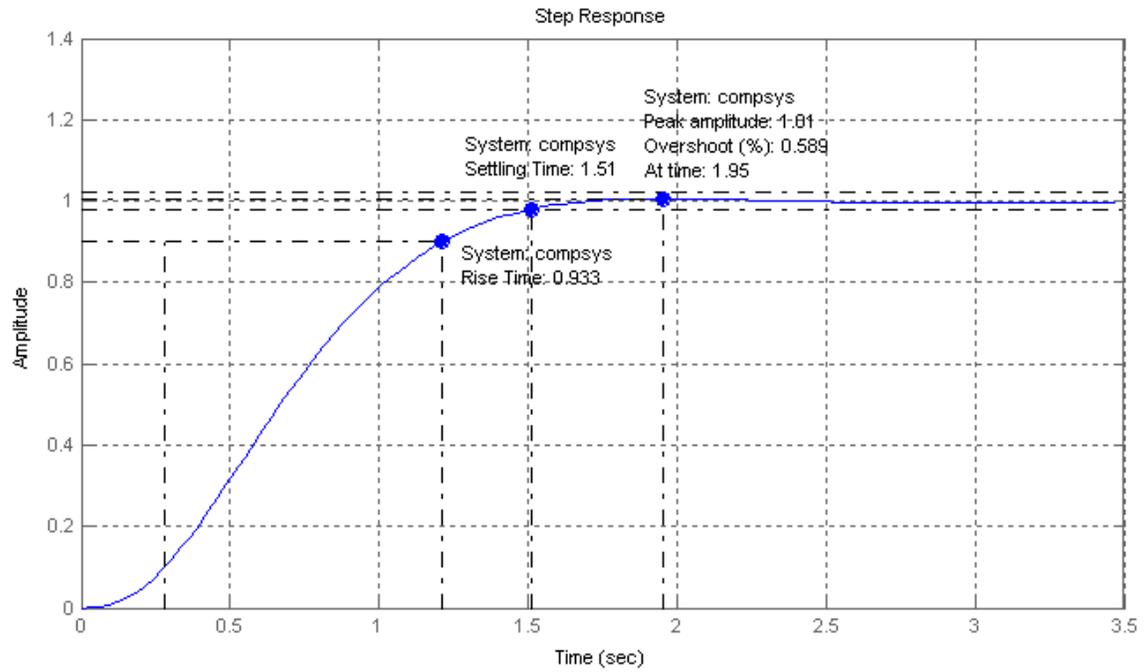The phase-lead controller is

$$G_c(s) = \frac{64(s+1.75)}{(s+8)}$$

The compensated open-loop transfer function is

$$G_c(s)GH(s) = \frac{64s+112}{s^4+15s^3+66s^2+80s}$$

and the closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{64s+112}{s^4+15s^3+66s^2+144s+112}$$

The compensated step response is as shown.

Step Response

You can use the rldesigngui to design the controllers for the remaining examples.