

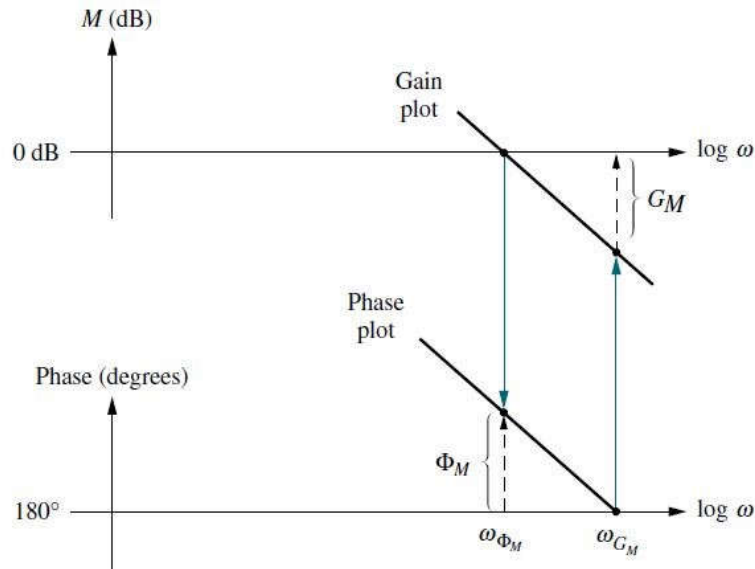
## LECTURE 4-Design of via Frequency Response-Lag Compensator

### 4.1 Frequency Response Techniques

- Frequency response methods are an alternative to the root locus for analyzing and designing feedback control systems.
- The input to a physical system can be sinusoidally varying with known frequency, amplitude, and phase angle. The system's output, which is also sinusoidal in the steady state, can then be measured for amplitude and phase angle at different frequencies. From this data the magnitude frequency response of the system, which is the ratio of the output amplitude to the input amplitude, can be plotted and used in place of an analytically obtained magnitude frequency response.
- Similarly, we can obtain the phase response by finding the difference between the output phase angle and the input phase angle at different frequencies.
- The frequency response of a system can be represented either as a polar plot or as separate magnitude and phase diagrams.
- As a polar plot, the magnitude response is the length of a vector drawn from the origin to a point on the curve, whereas the phase response is the angle of that vector.
- The polar plot of  $G(s)H(s)$  is known as a **Nyquist diagram**.
- Separate magnitude and phase diagrams, sometimes referred to as **Bode plots**. The magnitude curve can be a plot of log-magnitude versus log-frequency. The other graph is a plot of phase angle versus log-frequency.
- An advantage of Bode plots over the Nyquist diagram is that they can easily be drawn using asymptotic approximations to the actual curve.
- Using the Nyquist criterion and Nyquist diagram, or the Nyquist criterion and Bode plots, we can determine a system's stability.
- **Determining Stability**
  - The stability of a system can be determined from Bode plots (please recall, we have already discussed Bode plots) by implementing the Nyquist stability criterion using two plots (log-magnitude versus frequency plot, phase versus frequency plot). From Bode log-magnitude plot, determine the value of gain that ensures that the magnitude is less than 0 dB (unity gain) at that frequency where the phase is  $\pm 180^\circ$ .
  - Thus, the Nyquist criterion using Bode plots tells us how to determine if a system is stable. The stability of a system in frequency response methods can be stated as follows:  
*An open-loop stable system is stable in closed-loop if the open-loop magnitude frequency response has a gain of less than 0 dB at the frequency where the phase frequency response is  $\pm 180^\circ$  (that is, at phase cross-over frequency).*

- **Evaluating Gain and Phase Margins**

- Gain margin is the amount that the gain of a system can be increased before instability occurs if the phase angle is constant at  $180^\circ$ .
- The gain margin is found by using the phase plot to find the frequency,  $\omega_{GM}$  (phase crossover frequency), where the phase angle is  $\pm 180^\circ$ . At this frequency, we look at the magnitude plot to determine the gain margin,  $G_M$ , which is the gain required to raise the magnitude curve to 0 dB.



**Figure 1:** Gain and phase margins on the Bode diagrams

- Phase margin is the amount that the phase angle can be changed before instability occurs if the gain is held at unity.
- The phase margin is found by using the magnitude curve to find the frequency,  $\omega_{\phi_M}$  (gain cross-over frequency), where the gain is 0 dB. On the phase curve at that frequency, the phase margin,  $\phi_M$ , is the difference between the phase value and  $180^\circ$ .

- **Relation between Damping ratio and Closed-Loop Frequency response**

Recall the magnitude of resonant peak ( $M_p$ ) and bandwidth ( $w_{BW}$ ) of a second-order system is directly related to the damping ratio ( $\xi$ ), and hence, the percentage overshoot. And we know that the percentage overshoot is given by

$$\%M_p = e^{-\left(\xi\pi/\sqrt{1-\xi^2}\right)} \times 100$$

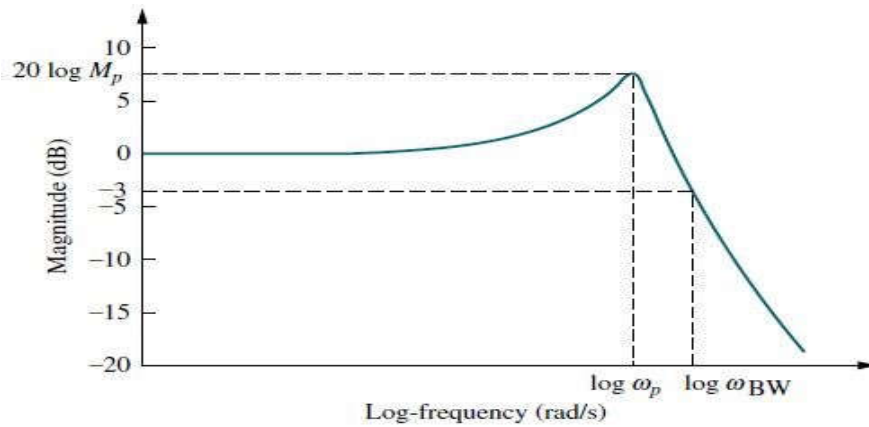
From the above equation, the damping ratio can be obtained as

$$\xi = \frac{-\ln(\%M_p/100)}{\sqrt{\pi^2 + \ln^2(\%M_p/100)}} \quad \text{(Equation (1))}$$

➤ Resonant peak is the maximum value of magnitude  $M$  and is given by,

$$M_p = \frac{1}{2\xi(1 - \xi^2)}$$

➤ And the frequency at which the resonant peak occurs is resonant frequency,  $\omega_p$  and is given  $\omega_p = \omega_n \sqrt{1 - 2\xi^2}$



**Figure 2:** Log-magnitude plot of second-order system

- **Relation between Transient response speed and Closed-Loop Frequency response**

Another relationship between the frequency response and time response is between the speed of the time response (as measured by settling time ( $T_s$ ), peak time ( $T_p$ ), and rise time ( $T_r$ )) and the **bandwidth** of the closed-loop frequency response, which is defined here as the frequency,  $\omega_{BW}$ , at which the magnitude response curve is 3 dB down from its value at zero frequency (see Figure 2).

Recall the equation,  $\omega_{BW} = \omega_n \sqrt{(1 - 2\xi^2) + \sqrt{4\xi^4 - 4\xi^2 + 2}}$  **(Equation (2))**

To relate  $\omega_{BW}$  to settling time, we substitute,  $\omega_n = 4/\xi T_s$  into Equation (2) and obtain

$$\omega_{BW} = \frac{4}{\xi T_s} \sqrt{(1 - 2\xi^2) + \sqrt{4\xi^4 - 4\xi^2 + 2}} \quad \text{(Equation (3))}$$

Similarly, to relate  $w_{BW}$  to peak time, we substitute  $w_n = \pi/(T_p)\sqrt{1-\xi^2}$  into Equation (2) and obtain

$$w_{WB} = \frac{4}{T_p\sqrt{1-\xi^2}}\sqrt{(1-2\xi^2) + \sqrt{4\xi^4 + 4\xi^2 + 2}} \quad \text{(Equation (4))}$$

- **Damping ratio from Phase margin**

The relationship between the phase margin and the damping ratio is given by (*without derivation*)

$$\phi_M = \tan^{-1}\left(\frac{2\xi}{\sqrt{-2\xi^2 + \sqrt{1+4\xi^4}}}\right) \quad \text{(Equation (5))}$$

This relationship will enable us to evaluate the percentage overshoot from the phase margin found from the open-loop frequency response.

- **Transient response design via cascade compensation**

- Recall with root locus, we can identify a specific point as having a desired transient response characteristic. We can then design cascade compensation to operate at that point and meet the transient response specifications.
- And from Equation (5) we seen that phase margin is related to percentage overshoot and from Equations (3) and (4) bandwidth is related to both damping ratio and settling time or peak time.
- When we design cascade compensation using frequency response methods to improve the transient response, we have to **reshape** the open-loop transfer function's frequency response to meet both the phase-margin requirement (percentage overshoot) and the bandwidth requirement (settling time or peak time).

- The following summary are the **important points to be remember** on compensator design via **frequency response methods**:

- (1) An open-loop stable system is stable in closed-loop if the open-loop magnitude frequency response has a gain of less than 0 dB at the frequency where the phase frequency response is  $\pm 180^\circ$ .
- (2) Percentage overshoot is reduced by increasing the phase margin, and the speed of the response is increased by increasing the bandwidth.
- (3) Steady-state error is improved by increasing the low-frequency magnitude responses.

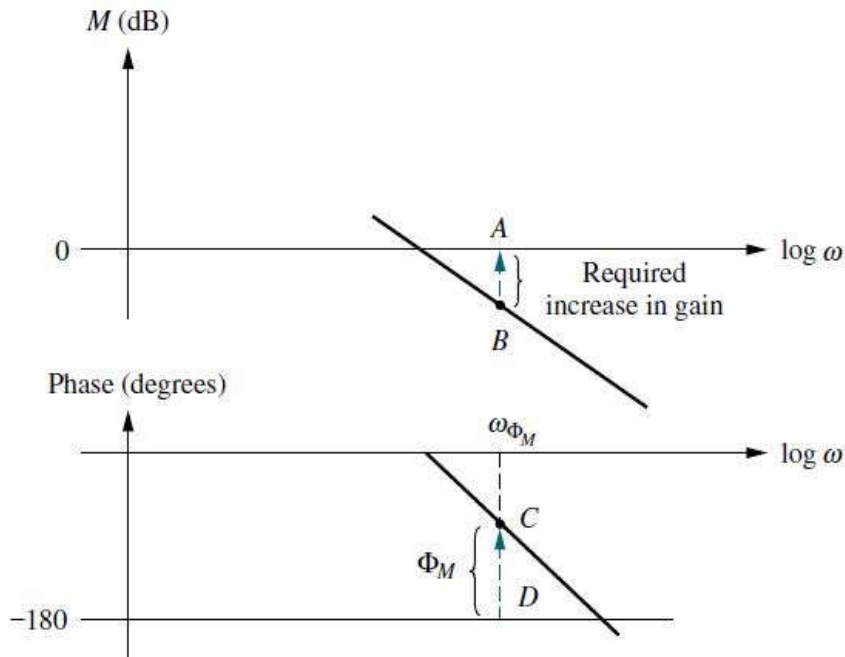
## 4.2 Transient Response via Gain Adjustment

Let us start with the design via frequency response methods by discussing the link between phase margin, transient response, and gain. In Equation (5), the relationship between damping ratio

(equivalently percentage overshoot) and phase margin was shown for  $G(s) = \frac{w_n^2}{s^2 + 2\xi w_n s}$ . Thus, if we

can vary the phase margin, we can vary the percentage overshoot. Looking at Figure 3, we see that if we desire a phase margin,  $\phi_M$  represented by  $CD$ , we would have to raise the magnitude curve by  $AB$ .

Thus, a simple gain adjustment can be used to design phase margin and, hence, percentage overshoot.



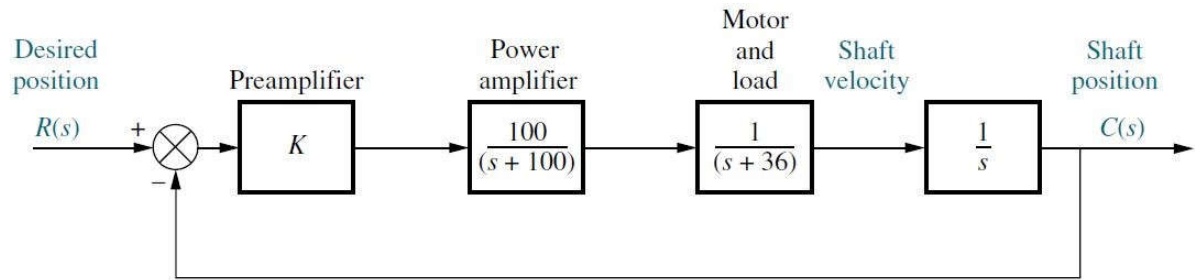
**Figure 3:** Bode plots showing gain adjustment for a desired phase margin

The following steps summarize the design procedure by which we can determine the gain to meet a percentage overshoot requirement using the open-loop frequency response and assuming dominant second-order closed-loop poles:

1. Draw the Bode magnitude and phase plots for a convenient value of gain.
2. Using Equations (1) and (5), determine the required phase margin from the percentage overshoot.
3. Find the frequency,  $w_{\phi_M}$  on the Bode phase diagram that yields the desired phase margin,  $CD$ , as shown on Figure 3.
4. Change the gain by an amount  $AB$  to force the magnitude curve to go through 0 dB at  $w_{\phi_M}$ . The amount of gain adjustment is the additional gain needed to produce the required phase margin.

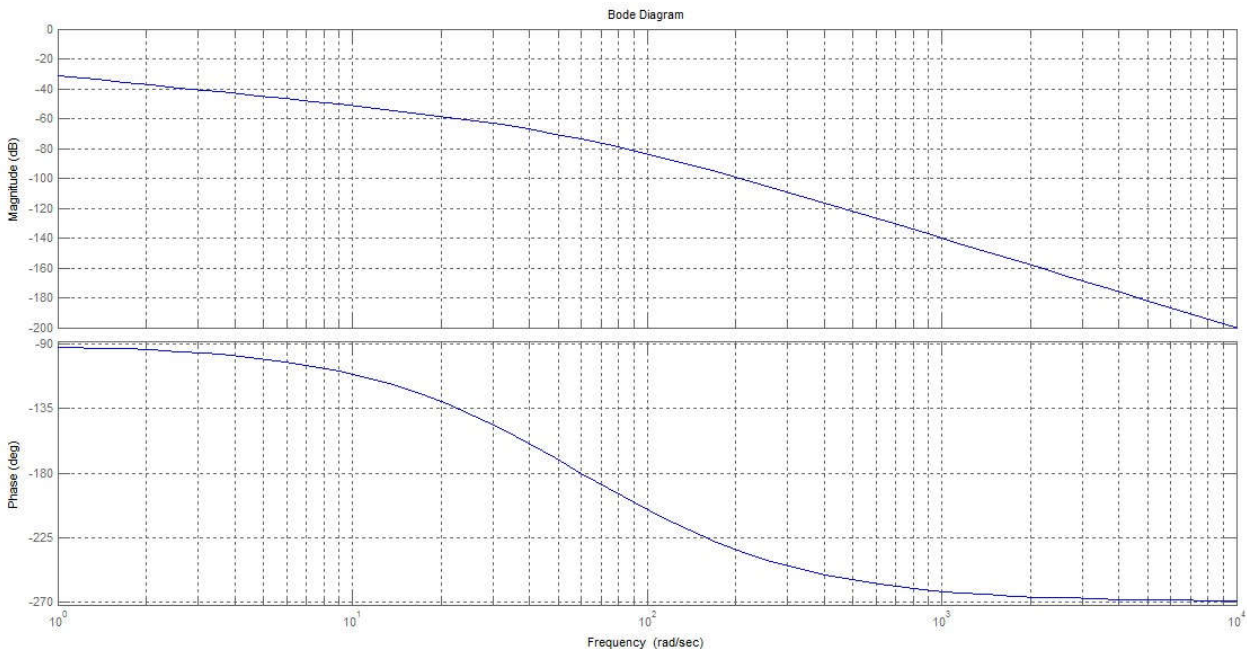
### Example 4.1: Transient Response via Gain Adjustment

**Problem:** For the position control system shown in Figure 4, find the value of preamplifier gain,  $K$ , to yield a 9.5% overshoot in the transient response for a step input using frequency response methods.



**Figure 4:** System for Example 1

**Step 1:** Choose  $K = 1$  (you can select any arbitrary value) to start the magnitude plot for open-loop transfer function by using a command 'bode' in MATLAB and the plots are shown in Figure 5.



**Figure 5:** Bode magnitude and phase plots for Example 1 for  $K = 1$

**Step 2:** Using Equation (1), a 9.5% overshoot implies  $\xi = 0.6$  for the closed-loop dominant poles.

Equation (5) yields a  $59.2^\circ$  phase margin for a damping ratio of 0.6.

**Step 3:** Locate on the phase plot the frequency that yields a  $59.2^\circ$  phase margin. This frequency is found where the phase angle is the difference between  $180^\circ$  and  $59.2^\circ$ , or  $-120.8^\circ$ . From the phase plot, the value of the phase-margin frequency is **14.76 rad/sec** (you can zoom the phase plot obtained from MATLAB (Figure 5) for finding the frequency for the phase value  $-120.8^\circ$ )

**Step 4:** At a frequency of 14.76 rad/sec on the magnitude plot, the gain is found to be **-55.3 dB** (you can zoom the magnitude plot obtained from MATLAB (Figure 5) for finding the dB value for the frequency 14.76 rad/sec). This magnitude has to be raised to 0 dB to yield the required phase margin.

The value of  $K$  for 55.3 dB can be obtained as

$$20\log_{10}K = 55.3 \quad \Rightarrow \log_{10}K = 2.765$$

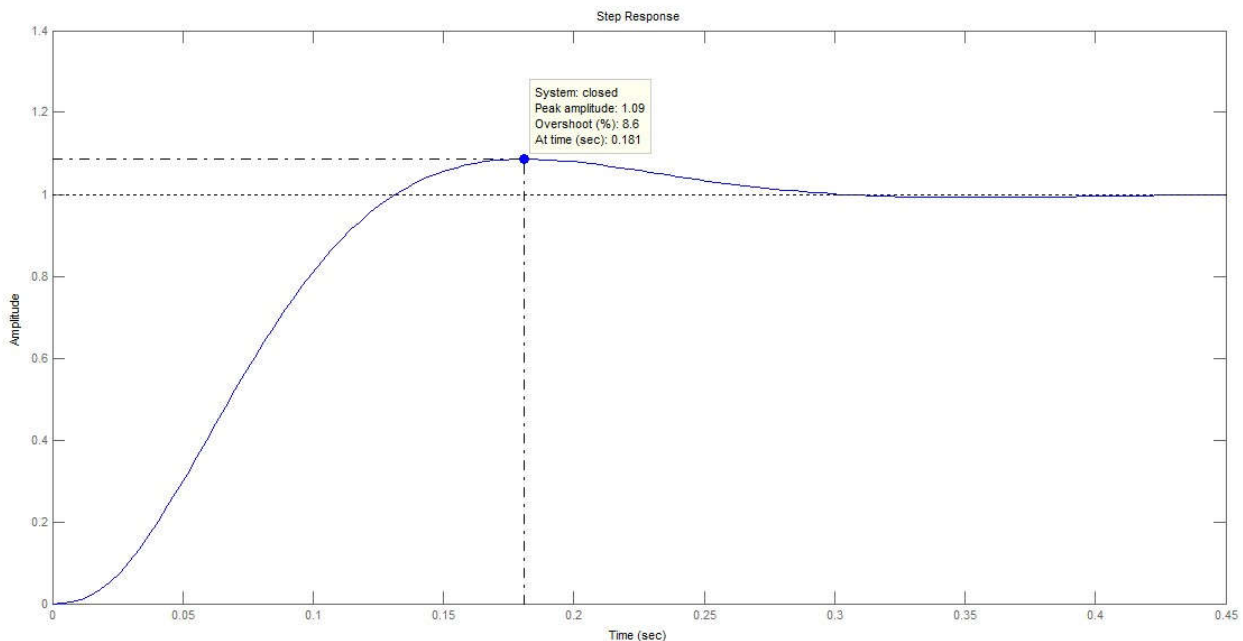
$$\Rightarrow K = 10^{2.765} = 582.1$$

Since the log-magnitude plot was drawn for  $K = 1$ , a 55.3 dB increase, or  $K = 1 \times 582.1 = \mathbf{582.1}$ , would yield the required phase margin for 9.5% overshoot.

Therefore, the gain-adjusted open-loop transfer function is

$$G(s) = \frac{100 \times 582.1}{s(s + 36)(s + 100)} = \frac{58210}{s(s + 36)(s + 100)}$$

Now, the open-loop transfer function can be checked for the given specification in MATLAB.



**Figure 6:** Step response of the system in Figure 4 for  $K = 582.1$

From Figure 6, it is clear that the percentage overshoot is 8.6% for  $K = 582.1$  (desired is 9.5%, so overdesign) and the peak time is 0.181 secs.

**Solving the problem in Example 1 shown in Figure 4 using MATLAB SISOTOOL:**

1. Type the following steps in MATLAB command window to determine the damping ratio for 9.5% overshoot:

$$ov = 9.5$$

$$damp = (-\log(ov/100)) / (\sqrt{\pi^2 + \log(ov/100)^2})$$

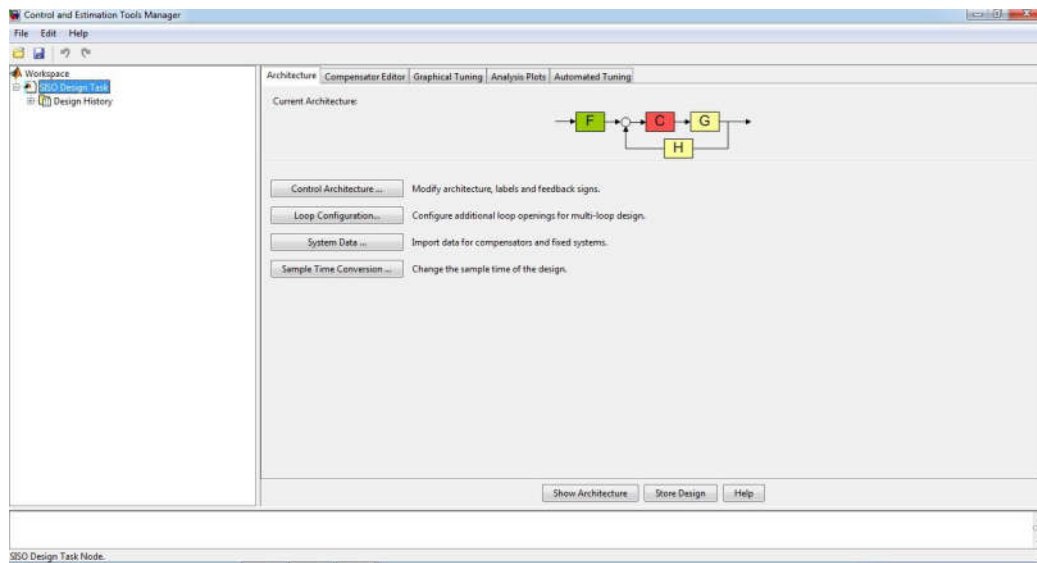
- Then, type the following step in MATLAB command window to determine the phase margin for the obtained damping ratio from the above step:

$$Pm = \text{atan}(2 * damp / (\sqrt{-2 * damp^2 + \sqrt{1 + 4 * damp^4}})) * (180 / \pi)$$

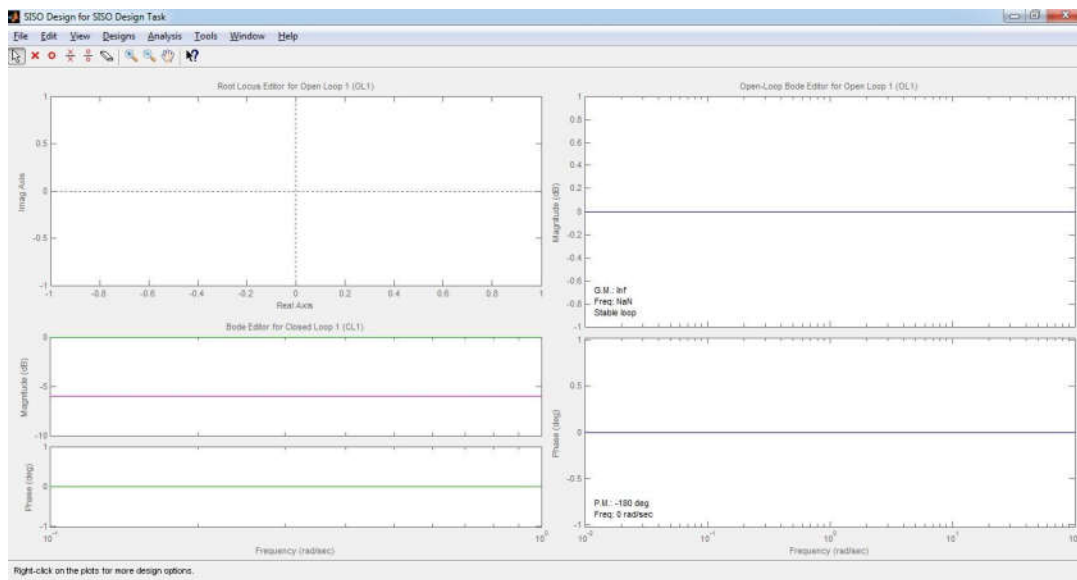
**Note:**

In MATLAB, ‘**log(x)**’ is the natural logarithm of x, and ‘**atan**’ is the inverse tangent and it returns the value in radians, therefore the value has to be converted into degrees by multiplying with 180/pi.

- Then, type `sisotool` to open SISO Design GUI. The following two windows (Window 1 (See Figure 7), Window 2 (See Figure 8)) appear:



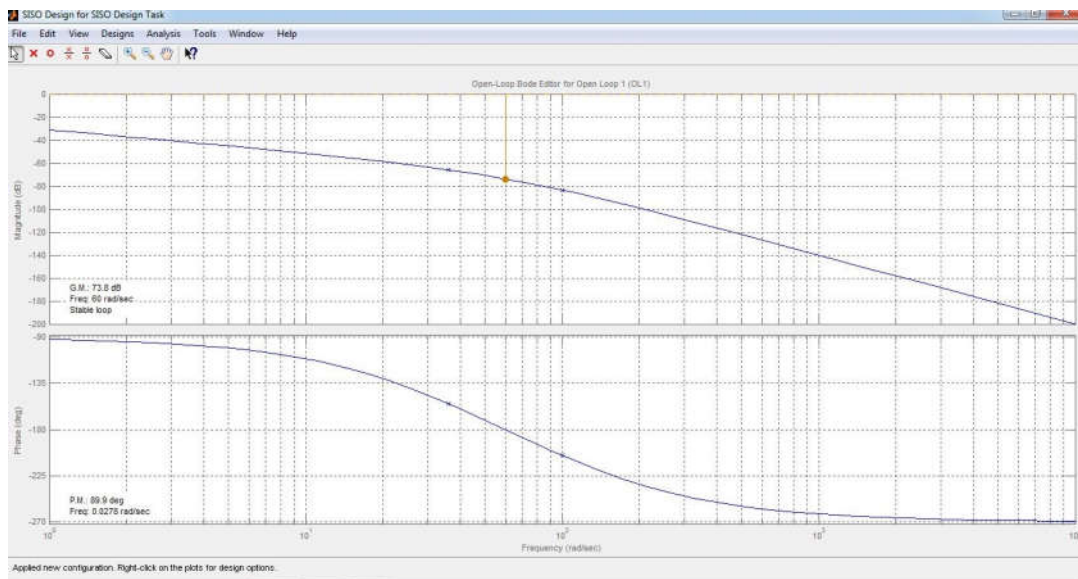
**Figure 7: SISOTOOL window 1 - ‘Control and Estimation Tools Manager’**



**Figure 8: SISOTOOL window 2 - ‘SISO Design for SISO Design Task’**



- In the SISOTOOL window **SISO Design for SISO Design Task**, Select **Design Plots Configuration...** in the **View** menu. In that, Select **Plot Type** as **Open-Loop Bode** for Plot 1 and Select **Plot Type** as **None** for Plots 2 and 3, and make sure that the **Plot Type** are **None** for all the plots except Plot 1.
- Then, go to **Architecture** tab in **Control and Estimation Tools Manager** window, and Click **System Data...** for import data for given open loop system. In that, the default value of 1 will be shown for G, H, C and F. Now, Double-click the **Data** for G to remove the value 1 and type the following syntax to load the system open-loop transfer function  $G(s) = \frac{100}{s^3+136s^2+3600}$  `tf(100,[1 136 3600 0])` and Click **OK** in the **System Data** window.
- Now, the Bode magnitude and phase plots for the given system G will be appeared in **SISO Design for SISO Design Task** window. In that, Right-click either inside Magnitude plot or inside Phase plot, and Select **Grid**. Now, grid appears in bode plots along with gain margin (G.M.) and phase margin (P.M.) values as shown below:



**Figure 9:** Bode Magnitude and Phase plots for the system,  $G(s)$  given in Figure 4

- Now, Grab the stability margin point in the magnitude diagram (indicated by orange circle) and raise the magnitude curve until the phase curve shows the phase margin calculated by the formula typed in MATLAB **Command window** and shown in the MATLAB **Command Window** as Pm (that is, 59.1621).
- After raising the magnitude curve for the phase margin value of 59.2 deg (shown in Phase plot), Right-click in the Bode plot area, Select **Edit Compensator...** and read the gain under

Compensator in the resulting window. That gain value is the value of open-loop  $K$  for the given system. Here, the value shown for Compensator is  $C = 584$  (Remember, the value is 582.1 with manual calculations)

Therefore, the gain-adjusted open-loop transfer function is

$$G(s) = \frac{100 \times 584}{s(s + 36)(s + 100)} = \frac{58400}{s^3 + 136s^2 + 3600s}$$

### 4.3 Lag Compensator Design

The following steps summarize the design procedure of lag compensator:

1. Set the gain,  $K$ , to the value that satisfies the steady-state error specification and plot the Bode magnitude and phase diagrams for this value of gain.
2. Find the frequency where the phase margin is  $5^\circ$  to  $12^\circ$  greater than the phase margin that yields the desired transient response (According to a textbook by Ogata (1990)). This step compensates for the fact that the phase of the lag compensator may still contribute anywhere from  $-5^\circ$  to  $-12^\circ$  of phase at the phase-margin frequency.

3. Select a lag compensator whose magnitude response yields a composite Bode magnitude diagram that goes through 0 dB at the frequency found in Step 2 as follows:

Draw the compensator's high-frequency asymptote to yield 0 dB for the compensated system at the frequency found in Step 2. Thus, if the gain at the frequency found in Step 2 is  $20 \log K_{PM}$ , then the compensator's high-frequency asymptote will be set at  $-20 \log K_{PM}$ ; select the upper break frequency to be 1 decade below the frequency found in Step 2; select the low-frequency asymptote to be at 0 dB; connect the compensator's high- and low-frequency asymptotes with a  $-20$  dB/decade line to locate the lower break frequency.

4. Reset the system gain,  $K$ , to compensate for any attenuation in the lag network in order to keep the static error constant the same as that found in Step 1.

From these steps, you see that we are relying upon the initial gain setting to meet the steady-state requirements and then relying upon the lag compensator's  $-20$  dB/decade slope to meet the transient response requirement by setting the 0 dB crossing of the magnitude plot.

#### **Example 4.2: Lag Compensator Design**

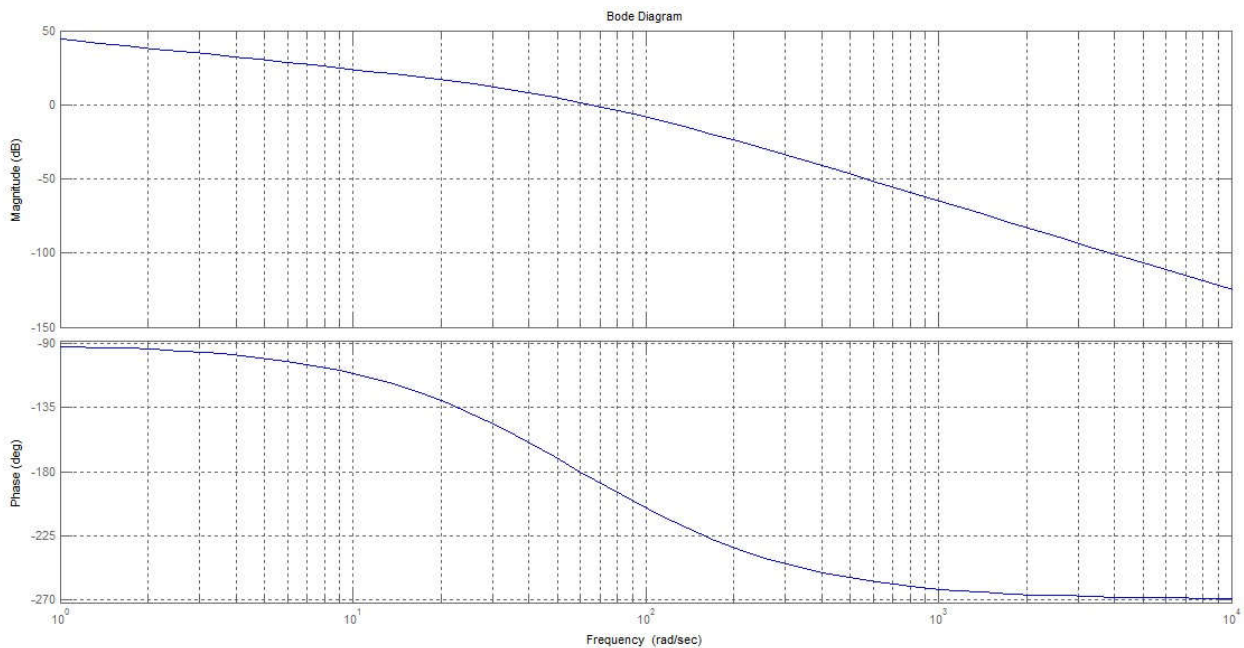
**Problem:** Given the system of Figure 4, use Bode diagrams to design a lag compensator to yield a tenfold improvement in steady-state error over the gain-compensated system while keeping the percentage overshoot at 9.5%.

**Step 1:** From Example 1 a gain,  $K$ , of 582.1 yields a 9.5% overshoot. Thus, for this system,  $K_v = 16.17$ . For a tenfold improvement in steady-state error,  $K_v$  must increase by a factor of 10, or  $K_v = 161.7$ . Therefore, the value of  $K$  in Figure 4 equals 5821, and the open-loop transfer function is

$$G(s) = \frac{582100}{s(s + 36)(s + 100)}$$

**Remember,**  $e(\infty) = \frac{1}{\lim_{s \rightarrow 0} sG(s)} = \frac{1}{K_v} \Rightarrow K_v = \lim_{s \rightarrow 0} sG(s)$  hence we can calculate for  $K_v$  if given  $G(s)$

The Bode plots for the open-loop transfer function for  $K = 5821$  are shown in Figure 10.

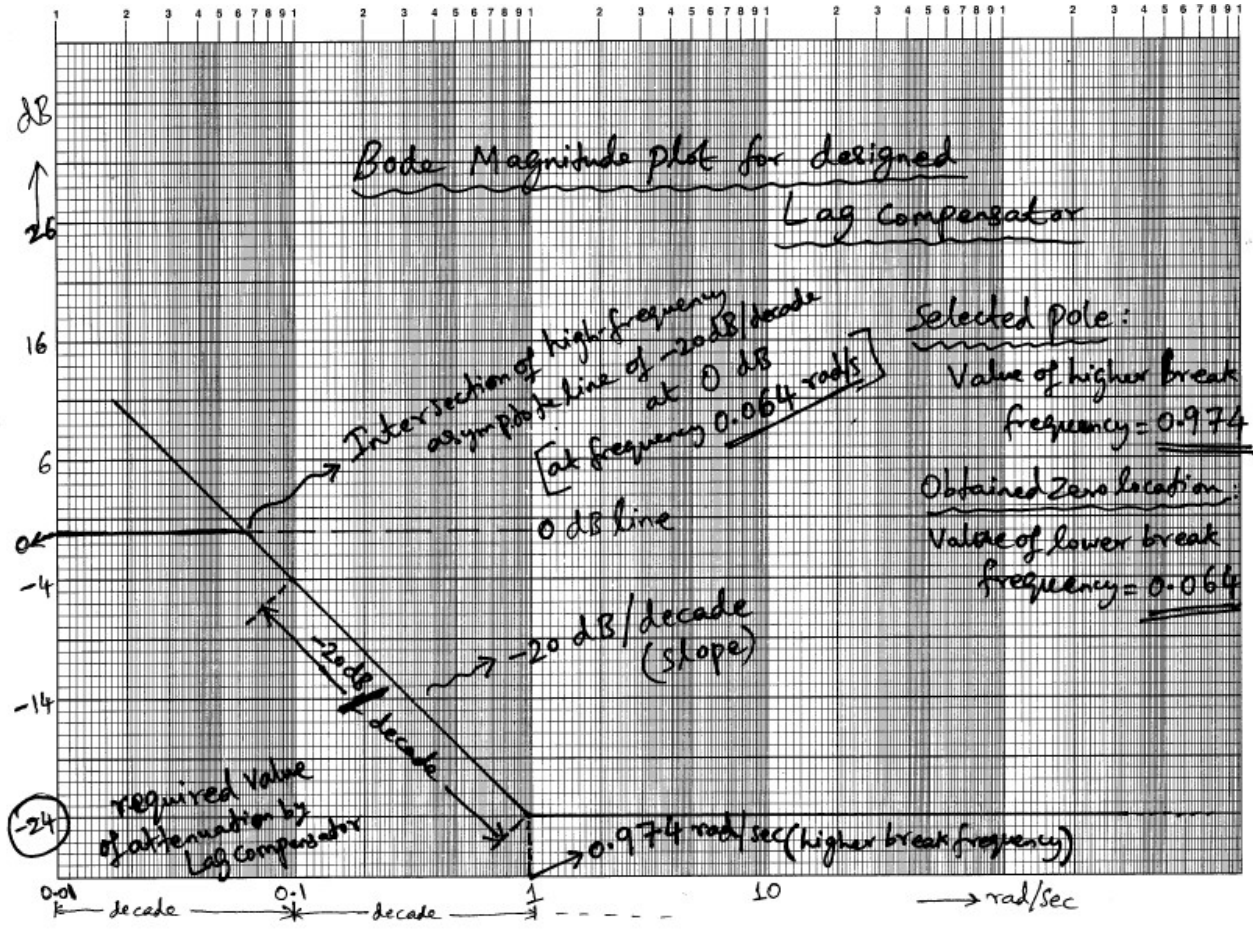


**Figure 10:** Bode magnitude and phase plots for Example 2 for  $K = 5821$

**Step 2:** The phase margin required for a 9.5% overshoot ( $\xi = 0.6$ ) is found from Equation (5) to be  $59.2^\circ$ . We increase this value of phase margin by  $10^\circ$  (according to Ogata, it is  $5^\circ$  to  $12^\circ$ ) to  $69.2^\circ$  in order to compensate for the phase angle contribution of the lag compensator. Now find the frequency where the phase margin is  $69.2^\circ$ . This frequency occurs at a phase angle of  $-180^\circ + 69.2^\circ = -110.8^\circ$  and is found to be **9.74 rad/s** (from Phase plot of Figure 10). At this frequency, the magnitude plot must go through 0dB. The magnitude at 9.74 rad/s is now  $+24.05 \text{ dB} \cong +24 \text{ dB}$  (from Magnitude plot of Figure 10). Thus, the lag compensator must provide **-24 dB attenuation at 9.74 rad/s**.

**Steps 3 and 4:** We now design the compensator. First draw the high-frequency asymptote at  $-24 \text{ dB}$ . Arbitrarily select the higher break frequency to be about one decade below the phase-margin frequency, or **0.974 rad/s**. Starting at the intersection of this frequency with the lag compensator's high-frequency

asymptote, draw a  $-20$  dB/decade line until  $0$  dB is reached. The compensator must have a dc gain of unity to retain the value of  $K_v$  that we have already designed by setting  $K = 5821$ . The lower break frequency is found to be **0.064 rad/s** (See Figure 11).



**Figure 11:** Manual procedure in Bode Magnitude plot for Steps 3 and 4

Now, the value of higher break frequency is taken as compensator zero, that is =  $0.974$  and the low-frequency value is the value of compensator pole and that is equal to  $0.064$ . And in Step 2, we see that the compensator should provide  $-24$  dB attenuation at  $9.74$  rad/s. Therefore, we have to find the gain value corresponds to  $-24$  dB as follows:

$$20\log K = -24 \Rightarrow \log K = -1.2 \Rightarrow K = 10^{-1.2} = 0.063$$

Hence, the lag compensator's transfer function is  $G_c(s) = \frac{0.063(s+0.974)}{(s+0.064)}$

Finally, the compensated system's forward transfer function is  $G(s)G_c(s) = \frac{36672.3(s+0.974)}{s(s+36)(s+100)(s+0.064)}$ .

The performance comparison of gain-compensated system and lag-compensated system are given in the below table (obtained from simulation results):

Parameter	Gain-compensated system	Lag-compensated system
Percentage overshoot	8.6%	9.1%
Steady-state error	0.0618	0.0065

#### Exercise 4.1 (for Practice)

Please try the following problem in MATLAB SISOTOOL:

For a unity feedback system with a forward transfer function  $G(s) = \frac{K}{s(s+50)(s+120)}$  use frequency response techniques to find the value of gain,  $K$ , to yield a closed-loop step response with 20% overshoot.

**ANSWER:**  $K = 194200$ .

#### Exercise 4.2 (for practice):

**Please try the following problem and verify using MATLAB:**

For a unity feedback system with a forward transfer function  $G(s) = \frac{K}{s(s+120)}$  use Bode

diagrams to design a lag compensator to yield a tenfold improvement in steady-state error over the gain compensated system while keeping the percentage overshoot at 20% (**Given in Exercise 4.1**).

**ANSWERS:**

Gain-compensated system to yield a tenfold improvement in steady-state error while maintaining 20% overshoot:

$$G(s) = \frac{194200}{s(s+50)(s+120)}$$

Lag-compensator to yield a tenfold improvement in steady-state error over the gain-compensated

$$\text{system: } G_c(s) = \frac{0.06863(s+2.035)}{(s+0.14)}$$

Steady-state error (for unit ramp input) for the gain-compensated system = 0.0309;

Steady-state error for the lag-compensated system = 0.0031.

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#### Reference textbook

Norman S. Nise, 'Control Systems Engineering', Sixth Edition, John Wiley and Sons, 2011.