3.13 SINGULARITY FUNCTIONS: STEPS, RAMPS, AND IMPULSES

In the study of control systems and the equations which describe them, a particular family of functions called *singularity functions* is used extensively. Each member of this family is related to the others by one or more integrations or differentiations. The three most widely used singularity functions are the *unit step*, the *unit impulse*, and the *unit ramp*.

Definition 3.15: A **unit step function** $\mathbf{1}(t-t_0)$ is defined by

$$\mathbf{1}(t-t_0)=\begin{cases}1 & \text{for } t>t_0\\ 0 & \text{for } t\le t_0\end{cases} \tag{3.17}$$

The unit step function is illustrated in Fig. 3-1.

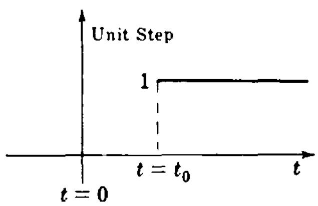

Fig. 3-1

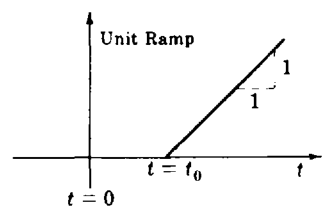

Fig. 3-2

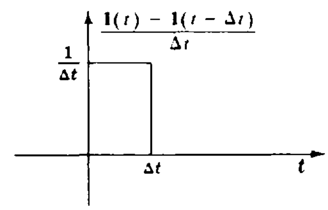

Fig. 3-3

Definition 3.16: A **unit ramp function** is the integral of a unit step function

$$\int_{-\infty}^{t}\mathbf{1}(\tau-t_0)\,d\tau=\begin{cases}t-t_0 & \text{for } t>t_0\\ 0 & \text{for } t\le t_0\end{cases} \tag{3.18}$$

The unit ramp function is illustrated in Fig. 3-2.

Definition 3.17: A **unit impulse function** $\delta(t)$ may be defined by

$$\delta(t)=\lim_{\substack{\Delta t\to 0\\ \Delta t>0}}\left[\frac{\mathbf{1}(t)-\mathbf{1}(t-\Delta t)}{\Delta t}\right] \tag{3.19*}$$

where $\mathbf{1}(t)$ is the unit step function.

The pair $\left\{\begin{matrix}\Delta t\to 0\\ \Delta t>0\end{matrix}\right\}$ may be abbreviated by $\Delta t\to 0^+$, meaning that Δt approaches zero *from the right*. The quotient in brackets represents a rectangle of height $1/\Delta t$ and width Δt as shown in Fig. 3-3. The limiting process produces a function whose height approaches infinity and width approaches zero. The area under the curve is equal to 1 for all values of Δt. That is,

$$\int_{-\infty}^{\infty}\delta(t)\,dt=1$$

The unit impulse function has the following very important property:

Screening Property: The integral of the product of a unit impulse function $\delta(t-t_0)$ and a function $f(t)$, continuous at $t=t_0$ over an interval which includes t_0, is equal to the function

*In a formal sense, Equation (3.19) defines the *one-sided derivative* of the unit step function. But neither the limit nor the derivative exist in the ordinary mathematical sense. However, Definition 3.17 is satisfactory for the purposes of this book, and many others.

$f(t)$ evaluated at t_0, that is,

$$\int_{-\infty}^{\infty} f(t)\,\delta(t-t_0)\,dt = f(t_0) \tag{3.20}$$

Definition 3.18: The **unit impulse response** of a system is the output $y(t)$ of the system when the input $u(t) = \delta(t)$ and all initial conditions are zero.

EXAMPLE 3.29. If the input-output relationship of a linear system is given by the convolution integral

$$y(t) = \int_0^t w(t-\tau)\,u(\tau)\,d\tau$$

then the unit impulse response $y_\delta(t)$ of the system is

$$y_\delta(t) = \int_0^t w(t-\tau)\,\delta(\tau)\,d\tau = \int_{-\infty}^{\infty} w(t-\tau)\,\delta(\tau)\,d\tau = w(t) \tag{3.21}$$

since $w(t-\tau) = 0$ for $\tau > t$, $\delta(\tau) = 0$ for $\tau < 0$, and the screening property of the unit impulse has been used to evaluate the integral.

Definition 3.19: The **unit step response** is the output $y(t)$ when the input $u(t) = \mathbf{1}(t)$ and all initial conditions are zero.

Definition 3.20: The **unit ramp response** is the output $y(t)$ when the input $u(t) = t$ for $t > 0$, $u(t) = 0$ for $t \leq 0$, and all initial conditions are zero.

3.14 SECOND-ORDER SYSTEMS

In the study of control systems, linear constant-coefficient second-order differential equations of the form:

$$\frac{d^2y}{dt^2} + 2\zeta\omega_n\frac{dy}{dt} + \omega_n^2 y = \omega_n^2 u \tag{3.22}$$

are important because higher-order systems can often be approximated by second-order systems. The constant ζ is called the **damping ratio**, and the constant ω_n is called the **undamped natural frequency** of the system. The forced response of this equation for inputs u belonging to the class of singularity functions is of particular interest. That is, the *forced response* to a unit impulse, unit step, or unit ramp is the same as the *unit impulse response*, *unit step response*, or *unit ramp response* of a system represented by this equation.

Assuming that $0 \leq \zeta \leq 1$, the characteristic equation for Equation (*3.22*) is

$$D^2 + 2\zeta\omega_n D + \omega_n^2 = \left(D + \zeta\omega_n - j\omega_n\sqrt{1-\zeta^2}\right)\left(D + \zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}\right) = 0$$

Hence the roots are

$$D_1 = -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2} \equiv -\alpha + j\omega_d \qquad D_2 = -\zeta\omega_n - j\omega_n\sqrt{1-\zeta^2} \equiv -\alpha - j\omega_d$$

where $\alpha \equiv \zeta\omega_n$ is called the **damping coefficient**, and $\omega_d \equiv \omega_n\sqrt{1-\zeta^2}$ is called the **damped natural frequency**. α is the inverse of the **time constant** τ of the system, that is, $\tau = 1/\alpha$.

The weighting function of Equation (*3.22*) is $w(t) = (1/\omega_d)e^{-\alpha t}\sin\omega_d t$. The unit step response is given by

$$y_1(t) = \int_0^t w(t-\tau)\,\omega_n^2\,d\tau = 1 - \frac{\omega_n e^{-\alpha t}}{\omega_d}\sin(\omega_d t + \phi) \tag{3.23}$$

where $\phi \equiv \tan^{-1}(\omega_d/\alpha)$.

Figure 3-4 is a parametric representation of the unit step response. Note that the abscissa of this family of curves is normalized time $\omega_n t$, and the parameter defining each curve is the damping ratio ζ.

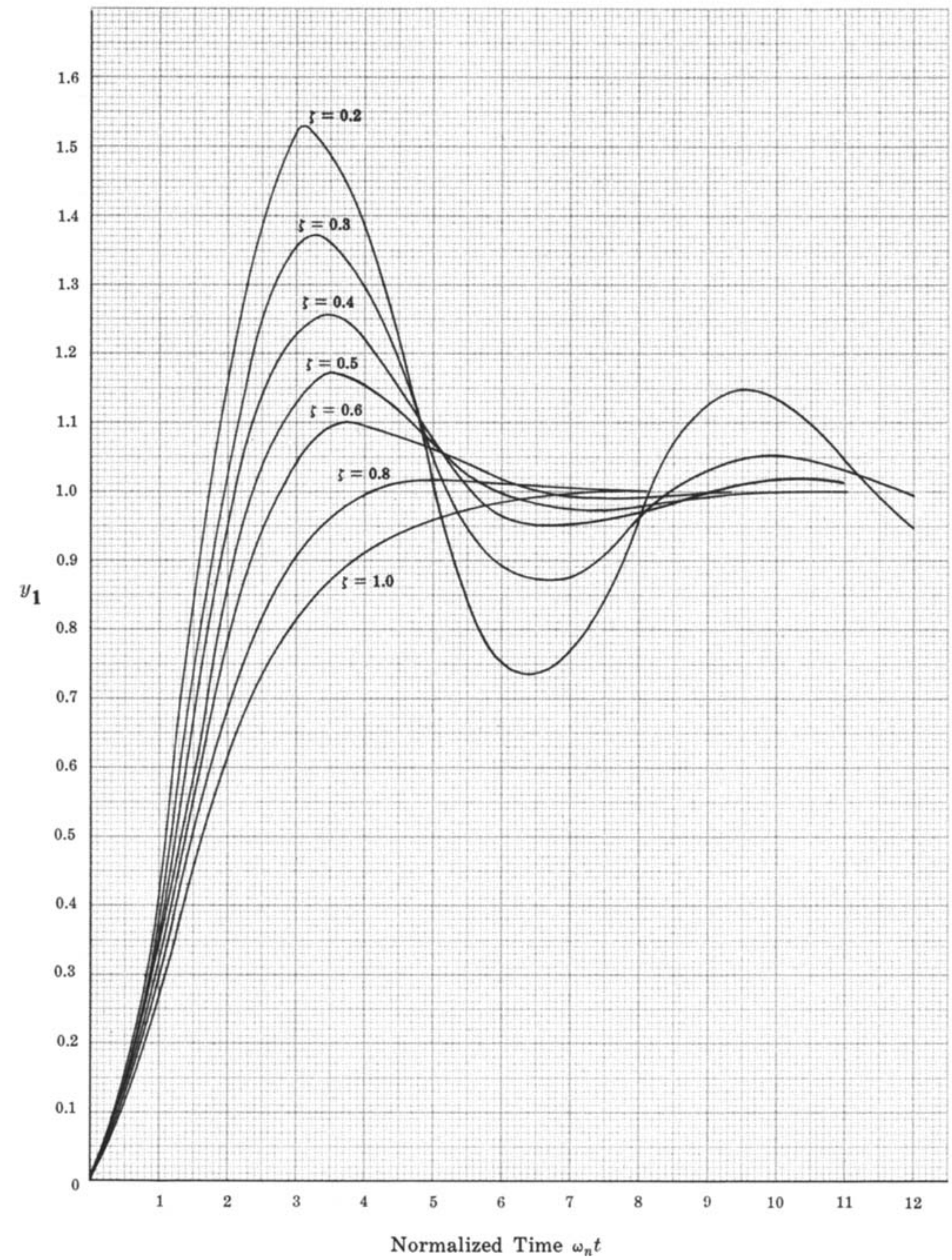

Fig. 3-4

3.15 STATE VARIABLE REPRESENTATION OF SYSTEMS DESCRIBED BY LINEAR DIFFERENTIAL EQUATIONS

In some problems of feedback and control, it is more convenient to describe a system by a set of first-order differential equations rather than by one or more nth-order differential equations. One reason is that quite general and powerful results from vector-matrix algebra can then be easily applied in deriving solutions for the differential equations.