



Control Systems Analysis and Design by the Root-Locus Method

6-1 INTRODUCTION

The basic characteristic of the transient response of a closed-loop system is closely related to the location of the closed-loop poles. If the system has a variable loop gain, then the location of the closed-loop poles depends on the value of the loop gain chosen. It is important, therefore, that the designer know how the closed-loop poles move in the s plane as the loop gain is varied.

From the design viewpoint, in some systems simple gain adjustment may move the closed-loop poles to desired locations. Then the design problem may become the selection of an appropriate gain value. If the gain adjustment alone does not yield a desired result, addition of a compensator to the system will become necessary. (This subject is discussed in detail in Sections 6-6 through 6-9.)

The closed-loop poles are the roots of the characteristic equation. Finding the roots of the characteristic equation of degree higher than 3 is laborious and will need computer solution. (MATLAB provides a simple solution to this problem.) However, just finding the roots of the characteristic equation may be of limited value, because as the gain of the open-loop transfer function varies, the characteristic equation changes and the computations must be repeated.

A simple method for finding the roots of the characteristic equation has been developed by W. R. Evans and used extensively in control engineering. This method, called the *root-locus method*, is one in which the roots of the characteristic equation

are plotted for all values of a system parameter. The roots corresponding to a particular value of this parameter can then be located on the resulting graph. Note that the parameter is usually the gain, but any other variable of the open-loop transfer function may be used. Unless otherwise stated, we shall assume that the gain of the open-loop transfer function is the parameter to be varied through all values, from zero to infinity.

By using the root-locus method the designer can predict the effects on the location of the closed-loop poles of varying the gain value or adding open-loop poles and/or open-loop zeros. Therefore, it is desired that the designer have a good understanding of the method for generating the root loci of the closed-loop system, both by hand and by use of a computer software program like MATLAB.

In designing a linear control system, we find that the root-locus method proves to be quite useful, since it indicates the manner in which the open-loop poles and zeros should be modified so that the response meets system performance specifications. This method is particularly suited to obtaining approximate results very quickly.

Because generating the root loci by use of MATLAB is very simple, one may think sketching the root loci by hand is a waste of time and effort. However, experience in sketching the root loci by hand is invaluable for interpreting computer-generated root loci, as well as for getting a rough idea of the root loci very quickly.

Outline of the Chapter. The outline of the chapter is as follows: Section 6–1 has presented an introduction to the root-locus method. Section 6–2 details the concepts underlying the root-locus method and presents the general procedure for sketching root loci using illustrative examples. Section 6–3 discusses generating root-locus plots with MATLAB. Section 6–4 treats a special case when the closed-loop system has positive feedback. Section 6–5 presents general aspects of the root-locus approach to the design of closed-loop systems. Section 6–6 discusses the control systems design by lead compensation. Section 6–7 treats the lag compensation technique. Section 6–8 deals with the lag–lead compensation technique. Finally, Section 6–9 discusses the parallel compensation technique.

6–2 ROOT-LOCUS PLOTS

Angle and Magnitude Conditions. Consider the negative feedback system shown in Figure 6–1. The closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} \quad (6-1)$$

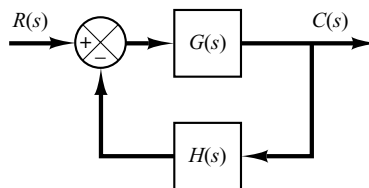


Figure 6–1
Control system.

The characteristic equation for this closed-loop system is obtained by setting the denominator of the right-hand side of Equation (6-1) equal to zero. That is,

$$1 + G(s)H(s) = 0$$

or

$$G(s)H(s) = -1 \quad (6-2)$$

Here we assume that $G(s)H(s)$ is a ratio of polynomials in s . [It is noted that we can extend the analysis to the case when $G(s)H(s)$ involves the transport lag e^{-Ts} .] Since $G(s)H(s)$ is a complex quantity, Equation (6-2) can be split into two equations by equating the angles and magnitudes of both sides, respectively, to obtain the following:

Angle condition:

$$\angle G(s)H(s) = \pm 180^\circ(2k + 1) \quad (k = 0, 1, 2, \dots) \quad (6-3)$$

Magnitude condition:

$$|G(s)H(s)| = 1 \quad (6-4)$$

The values of s that fulfill both the angle and magnitude conditions are the roots of the characteristic equation, or the closed-loop poles. A locus of the points in the complex plane satisfying the angle condition alone is the root locus. The roots of the characteristic equation (the closed-loop poles) corresponding to a given value of the gain can be determined from the magnitude condition. The details of applying the angle and magnitude conditions to obtain the closed-loop poles are presented later in this section.

In many cases, $G(s)H(s)$ involves a gain parameter K , and the characteristic equation may be written as

$$1 + \frac{K(s + z_1)(s + z_2) \cdots (s + z_m)}{(s + p_1)(s + p_2) \cdots (s + p_n)} = 0$$

Then the root loci for the system are the loci of the closed-loop poles as the gain K is varied from zero to infinity.

Note that to begin sketching the root loci of a system by the root-locus method we must know the location of the poles and zeros of $G(s)H(s)$. Remember that the angles of the complex quantities originating from the open-loop poles and open-loop zeros to the test point s are measured in the counterclockwise direction. For example, if $G(s)H(s)$ is given by

$$G(s)H(s) = \frac{K(s + z_1)}{(s + p_1)(s + p_2)(s + p_3)(s + p_4)}$$

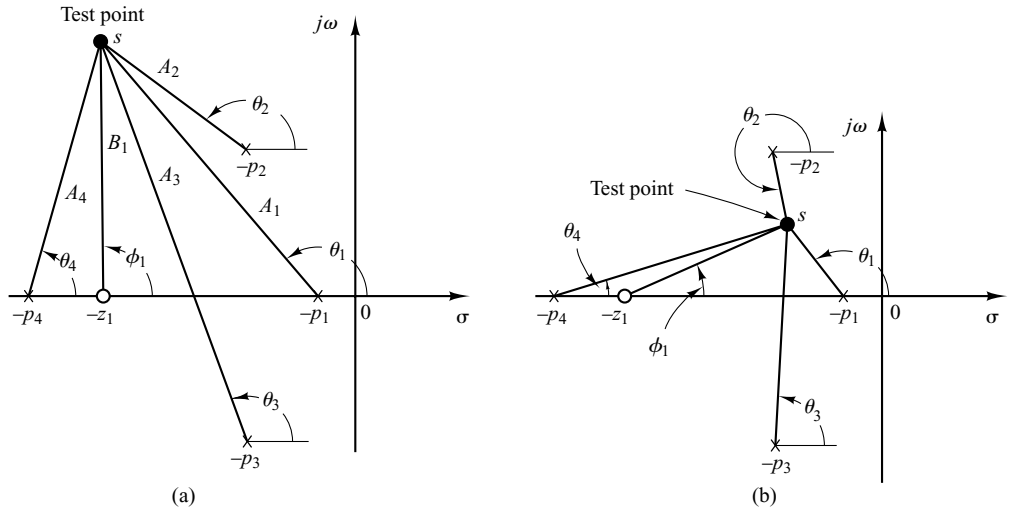


Figure 6-2
 (a) and (b) Diagrams showing angle measurements from open-loop poles and open-loop zero to test point s .

where $-p_2$ and $-p_3$ are complex-conjugate poles, then the angle of $G(s)H(s)$ is

$$\angle G(s)H(s) = \phi_1 - \theta_1 - \theta_2 - \theta_3 - \theta_4$$

where $\phi_1, \theta_1, \theta_2, \theta_3,$ and θ_4 are measured counterclockwise as shown in Figures 6-2(a) and (b). The magnitude of $G(s)H(s)$ for this system is

$$|G(s)H(s)| = \frac{KB_1}{A_1A_2A_3A_4}$$

where $A_1, A_2, A_3, A_4,$ and B_1 are the magnitudes of the complex quantities $s + p_1, s + p_2, s + p_3, s + p_4,$ and $s + z_1,$ respectively, as shown in Figure 6-2(a).

Note that, because the open-loop complex-conjugate poles and complex-conjugate zeros, if any, are always located symmetrically about the real axis, the root loci are always symmetrical with respect to this axis. Therefore, we only need to construct the upper half of the root loci and draw the mirror image of the upper half in the lower-half s plane.

Illustrative Examples. In what follows, two illustrative examples for constructing root-locus plots will be presented. Although computer approaches to the construction of the root loci are easily available, here we shall use graphical computation, combined with inspection, to determine the root loci upon which the roots of the characteristic equation of the closed-loop system must lie. Such a graphical approach will enhance understanding of how the closed-loop poles move in the complex plane as the open-loop poles and zeros are moved. Although we employ only simple systems for illustrative purposes, the procedure for finding the root loci is no more complicated for higher-order systems.

Because graphical measurements of angles and magnitudes are involved in the analysis, we find it necessary to use the same divisions on the abscissa as on the ordinate axis when sketching the root locus on graph paper.

EXAMPLE 6–1 Consider the negative feedback system shown in Figure 6–3. (We assume that the value of gain K is nonnegative.) For this system,

$$G(s) = \frac{K}{s(s+1)(s+2)}, \quad H(s) = 1$$

Let us sketch the root-locus plot and then determine the value of K such that the damping ratio ζ of a pair of dominant complex-conjugate closed-loop poles is 0.5.

For the given system, the angle condition becomes

$$\begin{aligned} \angle G(s) &= \angle \frac{K}{s(s+1)(s+2)} \\ &= -\angle s - \angle s+1 - \angle s+2 \\ &= \pm 180^\circ(2k+1) \quad (k = 0, 1, 2, \dots) \end{aligned}$$

The magnitude condition is

$$|G(s)| = \left| \frac{K}{s(s+1)(s+2)} \right| = 1$$

A typical procedure for sketching the root-locus plot is as follows:

1. Determine the root loci on the real axis. The first step in constructing a root-locus plot is to locate the open-loop poles, $s = 0$, $s = -1$, and $s = -2$, in the complex plane. (There are no open-loop zeros in this system.) The locations of the open-loop poles are indicated by crosses. (The locations of the open-loop zeros in this book will be indicated by small circles.) Note that the starting points of the root loci (the points corresponding to $K = 0$) are open-loop poles. The number of individual root loci for this system is three, which is the same as the number of open-loop poles.

To determine the root loci on the real axis, we select a test point, s . If the test point is on the positive real axis, then

$$\angle s = \angle s+1 = \angle s+2 = 0^\circ$$

This shows that the angle condition cannot be satisfied. Hence, there is no root locus on the positive real axis. Next, select a test point on the negative real axis between 0 and -1 . Then

$$\angle s = 180^\circ, \quad \angle s+1 = \angle s+2 = 0^\circ$$

Thus

$$-\angle s - \angle s+1 - \angle s+2 = -180^\circ$$

and the angle condition is satisfied. Therefore, the portion of the negative real axis between 0 and -1 forms a portion of the root locus. If a test point is selected between -1 and -2 , then

$$\angle s = \angle s+1 = 180^\circ, \quad \angle s+2 = 0^\circ$$

and

$$-\angle s - \angle s+1 - \angle s+2 = -360^\circ$$

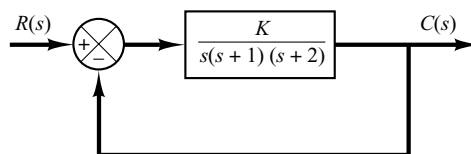


Figure 6–3
Control system.

It can be seen that the angle condition is not satisfied. Therefore, the negative real axis from -1 to -2 is not a part of the root locus. Similarly, if a test point is located on the negative real axis from -2 to $-\infty$, the angle condition is satisfied. Thus, root loci exist on the negative real axis between 0 and -1 and between -2 and $-\infty$.

2. Determine the asymptotes of the root loci. The asymptotes of the root loci as s approaches infinity can be determined as follows: If a test point s is selected very far from the origin, then

$$\lim_{s \rightarrow \infty} G(s) = \lim_{s \rightarrow \infty} \frac{K}{s(s+1)(s+2)} = \lim_{s \rightarrow \infty} \frac{K}{s^3}$$

and the angle condition becomes

$$-3 \angle s = \pm 180^\circ(2k + 1) \quad (k = 0, 1, 2, \dots)$$

or

$$\text{Angles of asymptotes} = \frac{\pm 180^\circ(2k + 1)}{3} \quad (k = 0, 1, 2, \dots)$$

Since the angle repeats itself as k is varied, the distinct angles for the asymptotes are determined as 60° , -60° , and 180° . Thus, there are three asymptotes. The one having the angle of 180° is the negative real axis.

Before we can draw these asymptotes in the complex plane, we must find the point where they intersect the real axis. Since

$$G(s) = \frac{K}{s(s+1)(s+2)}$$

if a test point is located very far from the origin, then $G(s)$ may be written as

$$G(s) = \frac{K}{s^3 + 3s^2 + \dots}$$

For large values of s , this last equation may be approximated by

$$G(s) \doteq \frac{K}{(s+1)^3} \quad (6-5)$$

A root-locus diagram of $G(s)$ given by Equation (6-5) consists of three straight lines. This can be seen as follows: The equation of the root locus is

$$\angle \frac{K}{(s+1)^3} = \pm 180^\circ(2k + 1)$$

or

$$-3 \angle s + 1 = \pm 180^\circ(2k + 1)$$

which can be written as

$$\angle s + 1 = \pm 60^\circ(2k + 1)$$

By substituting $s = \sigma + j\omega$ into this last equation, we obtain

$$\angle \sigma + j\omega + 1 = \pm 60^\circ(2k + 1)$$

or

$$\tan^{-1} \frac{\omega}{\sigma + 1} = 60^\circ, \quad -60^\circ, \quad 0^\circ$$

Taking the tangent of both sides of this last equation,

$$\frac{\omega}{\sigma + 1} = \sqrt{3}, \quad -\sqrt{3}, \quad 0$$

which can be written as

$$\sigma + 1 - \frac{\omega}{\sqrt{3}} = 0, \quad \sigma + 1 + \frac{\omega}{\sqrt{3}} = 0, \quad \omega = 0$$

These three equations represent three straight lines, as shown in Figure 6–4. The three straight lines shown are the asymptotes. They meet at point $s = -1$. Thus, the abscissa of the intersection of the asymptotes and the real axis is obtained by setting the denominator of the right-hand side of Equation (6–5) equal to zero and solving for s . The asymptotes are almost parts of the root loci in regions very far from the origin.

3. Determine the breakaway point. To plot root loci accurately, we must find the breakaway point, where the root-locus branches originating from the poles at 0 and -1 break away (as K is increased) from the real axis and move into the complex plane. The breakaway point corresponds to a point in the s plane where multiple roots of the characteristic equation occur.

A simple method for finding the breakaway point is available. We shall present this method in the following: Let us write the characteristic equation as

$$f(s) = B(s) + KA(s) = 0 \tag{6-6}$$

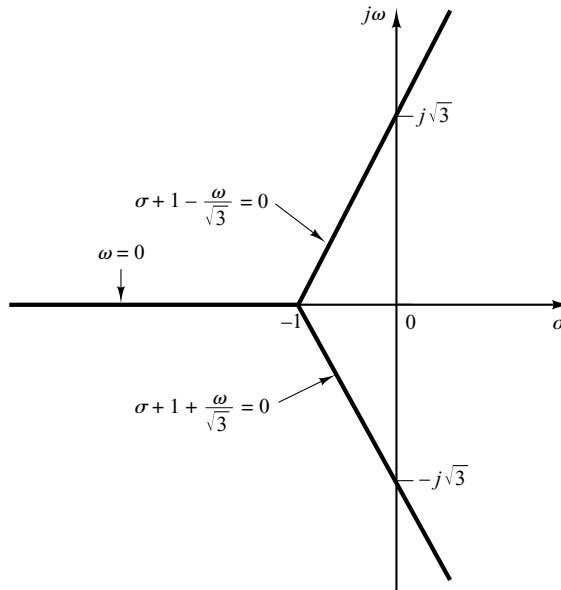


Figure 6–4
Three asymptotes.

where $A(s)$ and $B(s)$ do not contain K . Note that $f(s) = 0$ has multiple roots at points where

$$\frac{df(s)}{ds} = 0$$

This can be seen as follows: Suppose that $f(s)$ has multiple roots of order r , where $r \geq 2$. Then $f(s)$ may be written as

$$f(s) = (s - s_1)^r (s - s_2) \cdots (s - s_n)$$

Now we differentiate this equation with respect to s and evaluate $df(s)/ds$ at $s = s_1$. Then we get

$$\left. \frac{df(s)}{ds} \right|_{s=s_1} = 0 \quad (6-7)$$

This means that multiple roots of $f(s)$ will satisfy Equation (6-7). From Equation (6-6), we obtain

$$\frac{df(s)}{ds} = B'(s) + KA'(s) = 0 \quad (6-8)$$

where

$$A'(s) = \frac{dA(s)}{ds}, \quad B'(s) = \frac{dB(s)}{ds}$$

The particular value of K that will yield multiple roots of the characteristic equation is obtained from Equation (6-8) as

$$K = -\frac{B'(s)}{A'(s)}$$

If we substitute this value of K into Equation (6-6), we get

$$f(s) = B(s) - \frac{B'(s)}{A'(s)} A(s) = 0$$

or

$$B(s)A'(s) - B'(s)A(s) = 0 \quad (6-9)$$

If Equation (6-9) is solved for s , the points where multiple roots occur can be obtained. On the other hand, from Equation (6-6) we obtain

$$K = -\frac{B(s)}{A(s)}$$

and

$$\frac{dK}{ds} = -\frac{B'(s)A(s) - B(s)A'(s)}{A^2(s)}$$

If dK/ds is set equal to zero, we get the same equation as Equation (6-9). Therefore, the breakaway points can be simply determined from the roots of

$$\frac{dK}{ds} = 0$$

It should be noted that not all the solutions of Equation (6-9) or of $dK/ds = 0$ correspond to actual breakaway points. If a point at which $dK/ds = 0$ is on a root locus, it is an actual breakaway or break-in point. Stated differently, if at a point at which $dK/ds = 0$ the value of K takes a real positive value, then that point is an actual breakaway or break-in point.

For the present example, the characteristic equation $G(s) + 1 = 0$ is given by

$$\frac{K}{s(s+1)(s+2)} + 1 = 0$$

or

$$K = -(s^3 + 3s^2 + 2s)$$

By setting $dK/ds = 0$, we obtain

$$\frac{dK}{ds} = -(3s^2 + 6s + 2) = 0$$

or

$$s = -0.4226, \quad s = -1.5774$$

Since the breakaway point must lie on a root locus between 0 and -1 , it is clear that $s = -0.4226$ corresponds to the actual breakaway point. Point $s = -1.5774$ is not on the root locus. Hence, this point is not an actual breakaway or break-in point. In fact, evaluation of the values of K corresponding to $s = -0.4226$ and $s = -1.5774$ yields

$$K = 0.3849, \quad \text{for } s = -0.4226$$

$$K = -0.3849, \quad \text{for } s = -1.5774$$

4. Determine the points where the root loci cross the imaginary axis. These points can be found by use of Routh's stability criterion as follows: Since the characteristic equation for the present system is

$$s^3 + 3s^2 + 2s + K = 0$$

the Routh array becomes

$$\begin{array}{ccc} s^3 & 1 & 2 \\ s^2 & 3 & K \\ s^1 & \frac{6-K}{3} & \\ s^0 & K & \end{array}$$

The value of K that makes the s^1 term in the first column equal zero is $K = 6$. The crossing points on the imaginary axis can then be found by solving the auxiliary equation obtained from the s^2 row; that is,

$$3s^2 + K = 3s^2 + 6 = 0$$

which yields

$$s = \pm j\sqrt{2}$$

The frequencies at the crossing points on the imaginary axis are thus $\omega = \pm\sqrt{2}$. The gain value corresponding to the crossing points is $K = 6$.

An alternative approach is to let $s = j\omega$ in the characteristic equation, equate both the real part and the imaginary part to zero, and then solve for ω and K . For the present system, the characteristic equation, with $s = j\omega$, is

$$(j\omega)^3 + 3(j\omega)^2 + 2(j\omega) + K = 0$$

or

$$(K - 3\omega^2) + j(2\omega - \omega^3) = 0$$

Equating both the real and imaginary parts of this last equation to zero, respectively, we obtain

$$K - 3\omega^2 = 0, \quad 2\omega - \omega^3 = 0$$

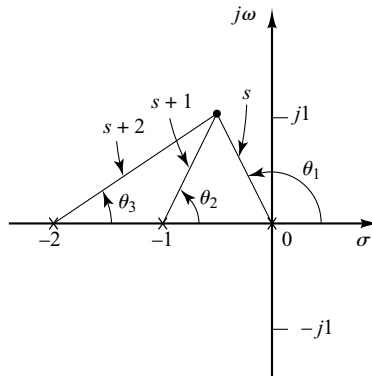


Figure 6-5
Construction of root locus.

from which

$$\omega = \pm\sqrt{2}, \quad K = 6 \quad \text{or} \quad \omega = 0, \quad K = 0$$

Thus, root loci cross the imaginary axis at $\omega = \pm\sqrt{2}$, and the value of K at the crossing points is 6. Also, a root-locus branch on the real axis touches the imaginary axis at $\omega = 0$. The value of K is zero at this point.

5. Choose a test point in the broad neighborhood of the $j\omega$ axis and the origin, as shown in Figure 6-5, and apply the angle condition. If a test point is on the root loci, then the sum of the three angles, $\theta_1 + \theta_2 + \theta_3$, must be 180° . If the test point does not satisfy the angle condition, select another test point until it satisfies the condition. (The sum of the angles at the test point will indicate the direction in which the test point should be moved.) Continue this process and locate a sufficient number of points satisfying the angle condition.

6. Draw the root loci, based on the information obtained in the foregoing steps, as shown in Figure 6-6.

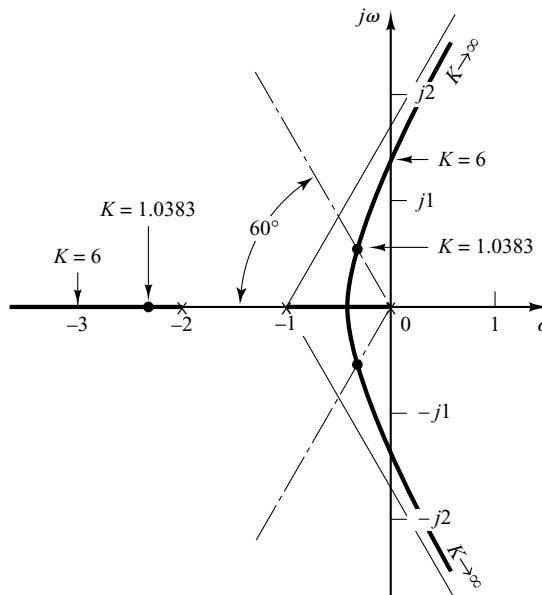


Figure 6-6
Root-locus plot.

7. Determine a pair of dominant complex-conjugate closed-loop poles such that the damping ratio ζ is 0.5. Closed-loop poles with $\zeta = 0.5$ lie on lines passing through the origin and making the angles $\pm\cos^{-1}\zeta = \pm\cos^{-1}0.5 = \pm60^\circ$ with the negative real axis. From Figure 6-6, such closed-loop poles having $\zeta = 0.5$ are obtained as follows:

$$s_1 = -0.3337 + j0.5780, \quad s_2 = -0.3337 - j0.5780$$

The value of K that yields such poles is found from the magnitude condition as follows:

$$\begin{aligned} K &= |s(s+1)(s+2)|_{s=-0.3337+j0.5780} \\ &= 1.0383 \end{aligned}$$

Using this value of K , the third pole is found at $s = -2.3326$.

Note that, from step 4, it can be seen that for $K = 6$ the dominant closed-loop poles lie on the imaginary axis at $s = \pm j\sqrt{2}$. With this value of K , the system will exhibit sustained oscillations. For $K > 6$, the dominant closed-loop poles lie in the right-half s plane, resulting in an unstable system.

Finally, note that, if necessary, the root loci can be easily graduated in terms of K by use of the magnitude condition. We simply pick out a point on a root locus, measure the magnitudes of the three complex quantities s , $s+1$, and $s+2$, and multiply these magnitudes; the product is equal to the gain value K at that point, or

$$|s| \cdot |s+1| \cdot |s+2| = K$$

Graduation of the root loci can be done easily by use of MATLAB. (See Section 6-3.)

EXAMPLE 6-2

In this example, we shall sketch the root-locus plot of a system with complex-conjugate open-loop poles. Consider the negative feedback system shown in Figure 6-7. For this system,

$$G(s) = \frac{K(s+2)}{s^2 + 2s + 3}, \quad H(s) = 1$$

where $K \geq 0$. It is seen that $G(s)$ has a pair of complex-conjugate poles at

$$s = -1 + j\sqrt{2}, \quad s = -1 - j\sqrt{2}$$

A typical procedure for sketching the root-locus plot is as follows:

1. *Determine the root loci on the real axis.* For any test point s on the real axis, the sum of the angular contributions of the complex-conjugate poles is 360° , as shown in Figure 6-8. Thus the net effect of the complex-conjugate poles is zero on the real axis. The location of the root locus on the real axis is determined from the open-loop zero on the negative real axis. A simple test reveals that a section of the negative real axis, that between -2 and $-\infty$, is a part of the root locus. It is noted that, since this locus lies between two zeros (at $s = -2$ and $s = -\infty$), it is actually a part of two root loci, each of which starts from one of the two complex-conjugate poles. In other words, two root loci break in the part of the negative real axis between -2 and $-\infty$.

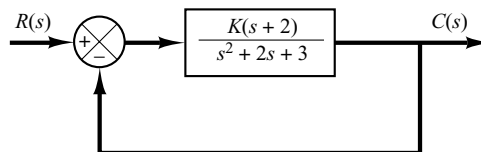


Figure 6-7
Control system.

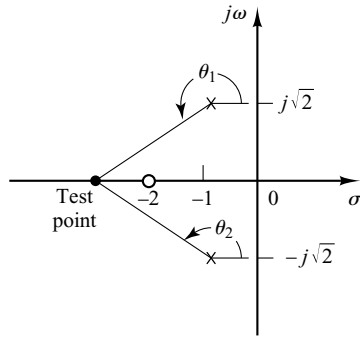


Figure 6–8
Determination of the root locus on the real axis.

Since there are two open-loop poles and one zero, there is one asymptote, which coincides with the negative real axis.

2. *Determine the angle of departure from the complex-conjugate open-loop poles.* The presence of a pair of complex-conjugate open-loop poles requires the determination of the angle of departure from these poles. Knowledge of this angle is important, since the root locus near a complex pole yields information as to whether the locus originating from the complex pole migrates toward the real axis or extends toward the asymptote.

Referring to Figure 6–9, if we choose a test point and move it in the very vicinity of the complex open-loop pole at $s = -p_1$, we find that the sum of the angular contributions from the pole at $s = p_2$ and zero at $s = -z_1$ to the test point can be considered remaining the same. If the test point is to be on the root locus, then the sum of ϕ'_1 , $-\theta_1$, and $-\theta'_2$ must be $\pm 180^\circ(2k + 1)$, where $k = 0, 1, 2, \dots$. Thus, in the example,

$$\phi'_1 - (\theta_1 + \theta'_2) = \pm 180^\circ(2k + 1)$$

or

$$\theta_1 = 180^\circ - \theta'_2 + \phi'_1 = 180^\circ - \theta_2 + \phi_1$$

The angle of departure is then

$$\theta_1 = 180^\circ - \theta_2 + \phi_1 = 180^\circ - 90^\circ + 55^\circ = 145^\circ$$

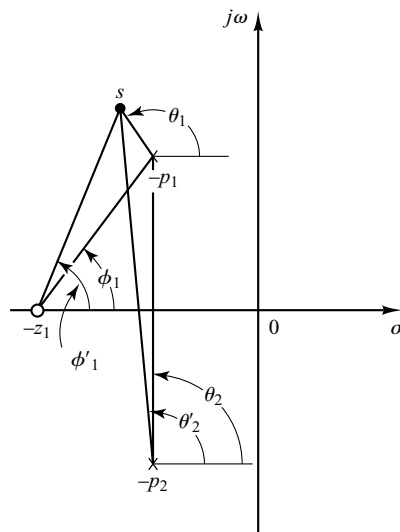


Figure 6–9
Determination of the angle of departure.

Since the root locus is symmetric about the real axis, the angle of departure from the pole at $s = -p_2$ is -145° .

3. *Determine the break-in point.* A break-in point exists where a pair of root-locus branches coalesces as K is increased. For this problem, the break-in point can be found as follows: Since

$$K = -\frac{s^2 + 2s + 3}{s + 2}$$

we have

$$\frac{dK}{ds} = -\frac{(2s + 2)(s + 2) - (s^2 + 2s + 3)}{(s + 2)^2} = 0$$

which gives

$$s^2 + 4s + 1 = 0$$

or

$$s = -3.7320 \quad \text{or} \quad s = -0.2680$$

Notice that point $s = -3.7320$ is on the root locus. Hence this point is an actual break-in point. (Note that at point $s = -3.7320$ the corresponding gain value is $K = 5.4641$.) Since point $s = -0.2680$ is not on the root locus, it cannot be a break-in point. (For point $s = -0.2680$, the corresponding gain value is $K = -1.4641$.)

4. *Sketch a root-locus plot, based on the information obtained in the foregoing steps.* To determine accurate root loci, several points must be found by trial and error between the break-in point and the complex open-loop poles. (To facilitate sketching the root-locus plot, we should find the direction in which the test point should be moved by mentally summing up the changes on the angles of the poles and zeros.) Figure 6–10 shows a complete root-locus plot for the system considered.

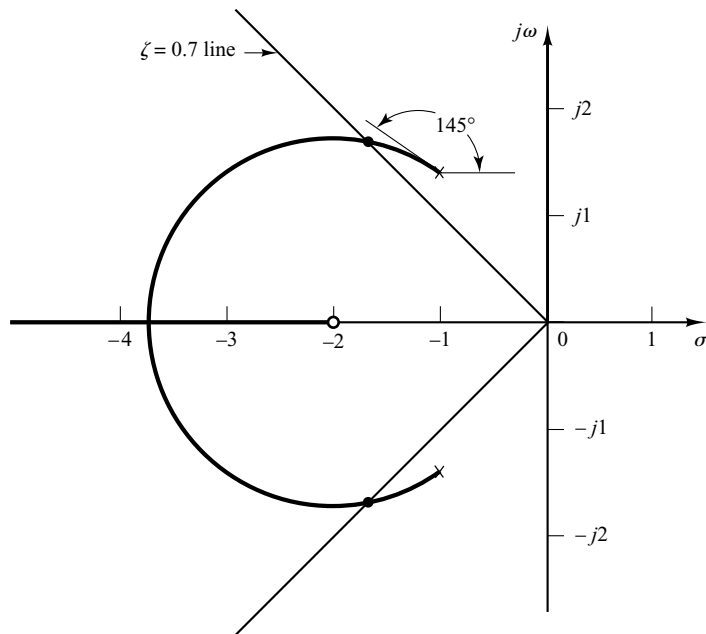


Figure 6–10
Root-locus plot.

The value of the gain K at any point on root locus can be found by applying the magnitude condition or by use of MATLAB (see Section 6–3). For example, the value of K at which the complex-conjugate closed-loop poles have the damping ratio $\zeta = 0.7$ can be found by locating the roots, as shown in Figure 6–10, and computing the value of K as follows:

$$K = \left| \frac{(s + 1 - j\sqrt{2})(s + 1 + j\sqrt{2})}{s + 2} \right|_{s=-1.67+j1.70} = 1.34$$

Or use MATLAB to find the value of K . (See Section 6–4.)

It is noted that in this system the root locus in the complex plane is a part of a circle. Such a circular root locus will not occur in most systems. Circular root loci may occur in systems that involve two poles and one zero, two poles and two zeros, or one pole and two zeros. Even in such systems, whether circular root loci occur depends on the locations of poles and zeros involved.

To show the occurrence of a circular root locus in the present system, we need to derive the equation for the root locus. For the present system, the angle condition is

$$\angle s + 2 - \angle s + 1 - j\sqrt{2} - \angle s + 1 + j\sqrt{2} = \pm 180^\circ(2k + 1)$$

If $s = \sigma + j\omega$ is substituted into this last equation, we obtain

$$\angle \sigma + 2 + j\omega - \angle \sigma + 1 + j\omega - j\sqrt{2} - \angle \sigma + 1 + j\omega + j\sqrt{2} = \pm 180^\circ(2k + 1)$$

which can be written as

$$\tan^{-1}\left(\frac{\omega}{\sigma + 2}\right) - \tan^{-1}\left(\frac{\omega - \sqrt{2}}{\sigma + 1}\right) - \tan^{-1}\left(\frac{\omega + \sqrt{2}}{\sigma + 1}\right) = \pm 180^\circ(2k + 1)$$

or

$$\tan^{-1}\left(\frac{\omega - \sqrt{2}}{\sigma + 1}\right) + \tan^{-1}\left(\frac{\omega + \sqrt{2}}{\sigma + 1}\right) = \tan^{-1}\left(\frac{\omega}{\sigma + 2}\right) \pm 180^\circ(2k + 1)$$

Taking tangents of both sides of this last equation using the relationship

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y} \quad (6-10)$$

we obtain

$$\tan\left[\tan^{-1}\left(\frac{\omega - \sqrt{2}}{\sigma + 1}\right) + \tan^{-1}\left(\frac{\omega + \sqrt{2}}{\sigma + 1}\right)\right] = \tan\left[\tan^{-1}\left(\frac{\omega}{\sigma + 2}\right) \pm 180^\circ(2k + 1)\right]$$

or

$$\frac{\frac{\omega - \sqrt{2}}{\sigma + 1} + \frac{\omega + \sqrt{2}}{\sigma + 1}}{1 - \left(\frac{\omega - \sqrt{2}}{\sigma + 1}\right)\left(\frac{\omega + \sqrt{2}}{\sigma + 1}\right)} = \frac{\frac{\omega}{\sigma + 2} \pm 0}{1 \mp \frac{\omega}{\sigma + 2} \times 0}$$

which can be simplified to

$$\frac{2\omega(\sigma + 1)}{(\sigma + 1)^2 - (\omega^2 - 2)} = \frac{\omega}{\sigma + 2}$$

or

$$\omega[(\sigma + 2)^2 + \omega^2 - 3] = 0$$

This last equation is equivalent to

$$\omega = 0 \quad \text{or} \quad (\sigma + 2)^2 + \omega^2 = (\sqrt{3})^2$$

These two equations are the equations for the root loci for the present system. Notice that the first equation, $\omega = 0$, is the equation for the real axis. The real axis from $s = -2$ to $s = -\infty$ corresponds to a root locus for $K \geq 0$. The remaining part of the real axis corresponds to a root locus when K is negative. (In the present system, K is nonnegative.) (Note that $K < 0$ corresponds to the positive-feedback case.) The second equation for the root locus is an equation of a circle with center at $\sigma = -2$, $\omega = 0$ and the radius equal to $\sqrt{3}$. That part of the circle to the left of the complex-conjugate poles corresponds to a root locus for $K \geq 0$. The remaining part of the circle corresponds to a root locus when K is negative.

It is important to note that easily interpretable equations for the root locus can be derived for simple systems only. For complicated systems having many poles and zeros, any attempt to derive equations for the root loci is discouraged. Such derived equations are very complicated and their configuration in the complex plane is difficult to visualize.

General Rules for Constructing Root Loci. For a complicated system with many open-loop poles and zeros, constructing a root-locus plot may seem complicated, but actually it is not difficult if the rules for constructing the root loci are applied. By locating particular points and asymptotes and by computing angles of departure from complex poles and angles of arrival at complex zeros, we can construct the general form of the root loci without difficulty.

We shall now summarize the general rules and procedure for constructing the root loci of the negative feedback control system shown in Figure 6–11.

First, obtain the characteristic equation

$$1 + G(s)H(s) = 0$$

Then rearrange this equation so that the parameter of interest appears as the multiplying factor in the form

$$1 + \frac{K(s + z_1)(s + z_2) \cdots (s + z_m)}{(s + p_1)(s + p_2) \cdots (s + p_n)} = 0 \quad (6-11)$$

In the present discussions, we assume that the parameter of interest is the gain K , where $K > 0$. (If $K < 0$, which corresponds to the positive-feedback case, the angle condition must be modified. See Section 6–4.) Note, however, that the method is still applicable to systems with parameters of interest other than gain. (See Section 6–6.)

1. *Locate the poles and zeros of $G(s)H(s)$ on the s plane. The root-locus branches start from open-loop poles and terminate at zeros (finite zeros or zeros at infinity). From the factored form of the open-loop transfer function, locate the open-loop poles and zeros in the s plane. [Note that the open-loop zeros are the zeros of $G(s)H(s)$, while the closed-loop zeros consist of the zeros of $G(s)$ and the poles of $H(s)$.]*

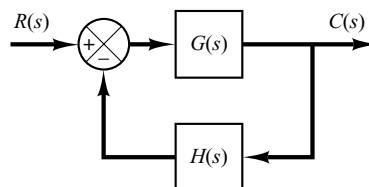


Figure 6–11
Control system.

Note that the root loci are symmetrical about the real axis of the s plane, because the complex poles and complex zeros occur only in conjugate pairs.

A root-locus plot will have just as many branches as there are roots of the characteristic equation. Since the number of open-loop poles generally exceeds that of zeros, the number of branches equals that of poles. If the number of closed-loop poles is the same as the number of open-loop poles, then the number of individual root-locus branches terminating at finite open-loop zeros is equal to the number m of the open-loop zeros. The remaining $n - m$ branches terminate at infinity ($n - m$ implicit zeros at infinity) along asymptotes.

If we include poles and zeros at infinity, the number of open-loop poles is equal to that of open-loop zeros. Hence we can always state that the root loci start at the poles of $G(s)H(s)$ and end at the zeros of $G(s)H(s)$, as K increases from zero to infinity, where the poles and zeros include both those in the finite s plane and those at infinity.

2. Determine the root loci on the real axis. Root loci on the real axis are determined by open-loop poles and zeros lying on it. The complex-conjugate poles and complex-conjugate zeros of the open-loop transfer function have no effect on the location of the root loci on the real axis because the angle contribution of a pair of complex-conjugate poles or complex-conjugate zeros is 360° on the real axis. Each portion of the root locus on the real axis extends over a range from a pole or zero to another pole or zero. In constructing the root loci on the real axis, choose a test point on it. If the total number of real poles and real zeros to the right of this test point is odd, then this point lies on a root locus. If the open-loop poles and open-loop zeros are simple poles and simple zeros, then the root locus and its complement form alternate segments along the real axis.

3. Determine the asymptotes of root loci. If the test point s is located far from the origin, then the angle of each complex quantity may be considered the same. One open-loop zero and one open-loop pole then cancel the effects of the other. Therefore, the root loci for very large values of s must be asymptotic to straight lines whose angles (slopes) are given by

$$\text{Angles of asymptotes} = \frac{\pm 180^\circ(2k + 1)}{n - m} \quad (k = 0, 1, 2, \dots)$$

where n = number of finite poles of $G(s)H(s)$

m = number of finite zeros of $G(s)H(s)$

Here, $k = 0$ corresponds to the asymptotes with the smallest angle with the real axis. Although k assumes an infinite number of values, as k is increased the angle repeats itself, and the number of distinct asymptotes is $n - m$.

All the asymptotes intersect at a point on the real axis. The point at which they do so is obtained as follows: If both the numerator and denominator of the open-loop transfer function are expanded, the result is

$$G(s)H(s) = \frac{K[s^m + (z_1 + z_2 + \dots + z_m)s^{m-1} + \dots + z_1 z_2 \dots z_m]}{s^n + (p_1 + p_2 + \dots + p_n)s^{n-1} + \dots + p_1 p_2 \dots p_n}$$

If a test point is located very far from the origin, then by dividing the denominator by the numerator, it is possible to write $G(s)H(s)$ as

$$G(s)H(s) = \frac{K}{s^{n-m} + [(p_1 + p_2 + \cdots + p_n) - (z_1 + z_2 + \cdots + z_m)]s^{n-m-1} + \cdots}$$

or

$$G(s)H(s) = \frac{K}{\left[s + \frac{(p_1 + p_2 + \cdots + p_n) - (z_1 + z_2 + \cdots + z_m)}{n - m} \right]^{n-m}} \quad (6-12)$$

The abscissa of the intersection of the asymptotes and the real axis is then obtained by setting the denominator of the right-hand side of Equation (6-12) equal to zero and solving for s , or

$$s = -\frac{(p_1 + p_2 + \cdots + p_n) - (z_1 + z_2 + \cdots + z_m)}{n - m} \quad (6-13)$$

[Example 6-1 shows why Equation (6-13) gives the intersection.] Once this intersection is determined, the asymptotes can be readily drawn in the complex plane.

It is important to note that the asymptotes show the behavior of the root loci for $|s| \gg 1$. A root-locus branch may lie on one side of the corresponding asymptote or may cross the corresponding asymptote from one side to the other side.

4. Find the breakaway and break-in points. Because of the conjugate symmetry of the root loci, the breakaway points and break-in points either lie on the real axis or occur in complex-conjugate pairs.

If a root locus lies between two adjacent open-loop poles on the real axis, then there exists at least one breakaway point between the two poles. Similarly, if the root locus lies between two adjacent zeros (one zero may be located at $-\infty$) on the real axis, then there always exists at least one break-in point between the two zeros. If the root locus lies between an open-loop pole and a zero (finite or infinite) on the real axis, then there may exist no breakaway or break-in points or there may exist both breakaway and break-in points.

Suppose that the characteristic equation is given by

$$B(s) + KA(s) = 0$$

The breakaway points and break-in points correspond to multiple roots of the characteristic equation. Hence, as discussed in Example 6-1, the breakaway and break-in points can be determined from the roots of

$$\frac{dK}{ds} = -\frac{B'(s)A(s) - B(s)A'(s)}{A^2(s)} = 0 \quad (6-14)$$

where the prime indicates differentiation with respect to s . It is important to note that the breakaway points and break-in points must be the roots of Equation (6-14), but not all roots of Equation (6-14) are breakaway or break-in points. If a real root of Equation (6-14) lies on the root-locus portion of the real axis, then it is an actual breakaway or break-in point. If a real root of Equation (6-14) is not on the root-locus portion of the real axis, then this root corresponds to neither a breakaway point nor a break-in point.

If two roots $s = s_1$ and $s = -s_1$ of Equation (6-14) are a complex-conjugate pair and if it is not certain whether they are on root loci, then it is necessary to check the corresponding K value. If the value of K corresponding to a root $s = s_1$ of $dK/ds = 0$ is positive, point $s = s_1$ is an actual breakaway or break-in point. (Since K is assumed to be nonnegative, if the value of K thus obtained is negative, or a complex quantity, then point $s = s_1$ is neither a breakaway nor a break-in point.)

5. Determine the angle of departure (angle of arrival) of the root locus from a complex pole (at a complex zero). To sketch the root loci with reasonable accuracy, we must find the directions of the root loci near the complex poles and zeros. If a test point is chosen and moved in the very vicinity of a complex pole (or complex zero), the sum of the angular contributions from all other poles and zeros can be considered to remain the same. Therefore, the angle of departure (or angle of arrival) of the root locus from a complex pole (or at a complex zero) can be found by subtracting from 180° the sum of all the angles of vectors from all other poles and zeros to the complex pole (or complex zero) in question, with appropriate signs included.

Angle of departure from a complex pole = 180°

– (sum of the angles of vectors to a complex pole in question from other poles)

+ (sum of the angles of vectors to a complex pole in question from zeros)

Angle of arrival at a complex zero = 180°

– (sum of the angles of vectors to a complex zero in question from other zeros)

+ (sum of the angles of vectors to a complex zero in question from poles)

The angle of departure is shown in Figure 6-12.

6. Find the points where the root loci may cross the imaginary axis. The points where the root loci intersect the $j\omega$ axis can be found easily by (a) use of Routh's stability criterion or (b) letting $s = j\omega$ in the characteristic equation, equating both the real part and the imaginary part to zero, and solving for ω and K . The values of ω thus found give the frequencies at which root loci cross the imaginary axis. The K value corresponding to each crossing frequency gives the gain at the crossing point.

7. Taking a series of test points in the broad neighborhood of the origin of the s plane, sketch the root loci. Determine the root loci in the broad neighborhood of the $j\omega$ axis and the origin. The most important part of the root loci is on neither the real axis nor the asymptotes but is in the broad neighborhood of the $j\omega$ axis and the origin. The shape

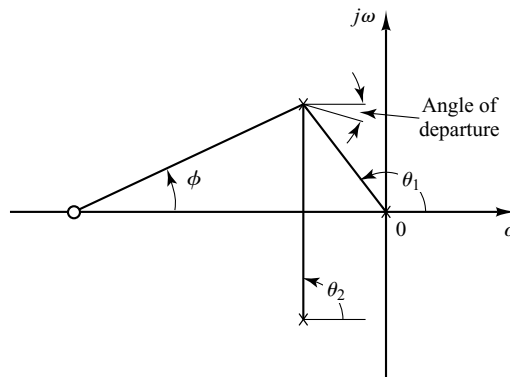


Figure 6-12
Construction of the root locus. [Angle of departure = $180^\circ - (\theta_1 + \theta_2) + \phi$.]

of the root loci in this important region in the s plane must be obtained with reasonable accuracy. (If accurate shape of the root loci is needed, MATLAB may be used rather than hand calculations of the exact shape of the root loci.)

8. Determine closed-loop poles. A particular point on each root-locus branch will be a closed-loop pole if the value of K at that point satisfies the magnitude condition. Conversely, the magnitude condition enables us to determine the value of the gain K at any specific root location on the locus. (If necessary, the root loci may be graduated in terms of K . The root loci are continuous with K .)

The value of K corresponding to any point s on a root locus can be obtained using the magnitude condition, or

$$K = \frac{\text{product of lengths between point } s \text{ to poles}}{\text{product of lengths between point } s \text{ to zeros}}$$

This value can be evaluated either graphically or analytically. (MATLAB can be used for graduating the root loci with K . See Section 6–3.)

If the gain K of the open-loop transfer function is given in the problem, then by applying the magnitude condition, we can find the correct locations of the closed-loop poles for a given K on each branch of the root loci by a trial-and-error approach or by use of MATLAB, which will be presented in Section 6–3.

Comments on the Root-Locus Plots. It is noted that the characteristic equation of the negative feedback control system whose open-loop transfer function is

$$G(s)H(s) = \frac{K(s^m + b_1s^{m-1} + \dots + b_m)}{s^n + a_1s^{n-1} + \dots + a_n} \quad (n \geq m)$$

is an n th-degree algebraic equation in s . If the order of the numerator of $G(s)H(s)$ is lower than that of the denominator by two or more (which means that there are two or more zeros at infinity), then the coefficient a_1 is the negative sum of the roots of the equation and is independent of K . In such a case, if some of the roots move on the locus toward the left as K is increased, then the other roots must move toward the right as K is increased. This information is helpful in finding the general shape of the root loci.

It is also noted that a slight change in the pole-zero configuration may cause significant changes in the root-locus configurations. Figure 6–13 demonstrates the fact that a slight change in the location of a zero or pole will make the root-locus configuration look quite different.

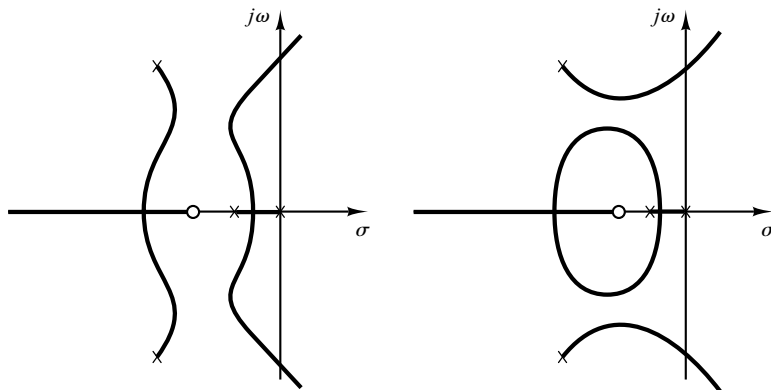


Figure 6–13
Root-locus plots.

Cancellation of Poles of $G(s)$ with Zeros of $H(s)$. It is important to note that if the denominator of $G(s)$ and the numerator of $H(s)$ involve common factors, then the corresponding open-loop poles and zeros will cancel each other, reducing the degree of the characteristic equation by one or more. For example, consider the system shown in Figure 6–14(a). (This system has velocity feedback.) By modifying the block diagram of Figure 6–14(a) to that shown in Figure 6–14(b), it is clearly seen that $G(s)$ and $H(s)$ have a common factor $s + 1$. The closed-loop transfer function $C(s)/R(s)$ is

$$\frac{C(s)}{R(s)} = \frac{K}{s(s + 1)(s + 2) + K(s + 1)}$$

The characteristic equation is

$$[s(s + 2) + K](s + 1) = 0$$

Because of the cancellation of the terms $(s + 1)$ appearing in $G(s)$ and $H(s)$, however, we have

$$\begin{aligned} 1 + G(s)H(s) &= 1 + \frac{K(s + 1)}{s(s + 1)(s + 2)} \\ &= \frac{s(s + 2) + K}{s(s + 2)} \end{aligned}$$

The reduced characteristic equation is

$$s(s + 2) + K = 0$$

The root-locus plot of $G(s)H(s)$ does not show all the roots of the characteristic equation, only the roots of the reduced equation.

To obtain the complete set of closed-loop poles, we must add the canceled pole of $G(s)H(s)$ to those closed-loop poles obtained from the root-locus plot of $G(s)H(s)$. The important thing to remember is that the canceled pole of $G(s)H(s)$ is a closed-loop pole of the system, as seen from Figure 6–14(c).

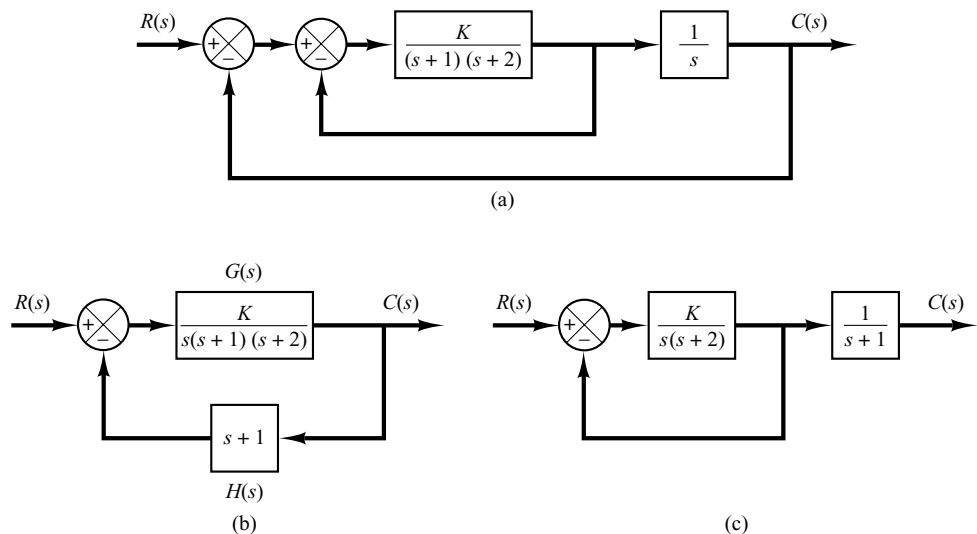


Figure 6–14
 (a) Control system with velocity feedback; (b) and (c) modified block diagrams.

Typical Pole–Zero Configurations and Corresponding Root Loci. In summarizing, we show several open-loop pole–zero configurations and their corresponding root loci in Table 6–1. The pattern of the root loci depends only on the relative separation of the open-loop poles and zeros. If the number of open-loop poles exceeds the number of finite zeros by three or more, there is a value of the gain K beyond which root loci enter the right-half s plane, and thus the system can become unstable. A stable system must have all its closed-loop poles in the left-half s plane.

Table 6–1 Open-Loop Pole–Zero Configurations and the Corresponding Root Loci

