

Example 2.3

Laplace Transform Solution of a Differential Equation

PROBLEM: Given the following differential equation, solve for $y(t)$ if all initial conditions are zero. Use the Laplace transform.

$$\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + 32y = 32u(t) \quad (2.14)$$

SOLUTION: Substitute the corresponding $F(s)$ for each term in Eq. (2.14), using Item 2 in Table 2.1, Items 7 and 8 in Table 2.2, and the initial conditions of $y(t)$ and $dy(t)/dt$ given by $y(0^-) = 0$ and $\dot{y}(0^-) = 0$, respectively. Hence, the Laplace transform of Eq. (2.14) is

$$s^2Y(s) + 12sY(s) + 32Y(s) = \frac{32}{s} \quad (2.15)$$

Solving for the response, $Y(s)$, yields

$$Y(s) = \frac{32}{s(s^2 + 12s + 32)} = \frac{32}{s(s+4)(s+8)} \quad (2.16)$$

To solve for $y(t)$, we notice that Eq. (2.16) does not match any of the terms in Table 2.1. Thus, we form the partial-fraction expansion of the right-hand term and match each of the resulting terms with $F(s)$ in Table 2.1. Therefore,

$$Y(s) = \frac{32 \mathbf{A}}{s(s+4)(s+8)} = \frac{\mathbf{K}_1}{s} + \frac{\mathbf{PD} \mathbf{K}_2}{(s+4)} + \frac{\mathbf{F} \mathbf{K}_3}{(s+8)} \quad (2.17)$$

where, from Eq. (2.13),

$$K_1 = \left. \frac{32}{(s+4)(s+8)} \right|_{s \rightarrow 0} = 1 \quad (2.18a)$$

$$K_2 = \left. \frac{32}{s(s+8)} \right|_{s \rightarrow -4} = -2 \quad (2.18b)$$

$$K_3 = \left. \frac{32}{s(s+4)} \right|_{s \rightarrow -8} = 1 \quad (2.18c)$$

Hence,

$$Y(s) = \frac{1}{s} - \frac{2}{(s+4)} + \frac{1}{(s+8)} \quad (2.19)$$

Since each of the three component parts of Eq. (2.19) is represented as an $F(s)$ in Table 2.1, $y(t)$ is the sum of the inverse Laplace transforms of each term. Hence,

$$y(t) = (1 - 2e^{-4t} + e^{-8t})u(t) \quad (2.20)$$

MATLAB
ML

Students who are using MATLAB should now run ch2p1 through ch2p8 in Appendix B. This is your first MATLAB exercise. You will learn how to use MATLAB to (1) represent polynomials, (2) find roots of polynomials, (3) multiply polynomials, and (4) find partial-fraction expansions. Finally, Example 2.3 will be solved using MATLAB.

TryIt 2.1

Use the following MATLAB and Control System Toolbox statement to form the linear, time-invariant (LTI) transfer function of Eq. (2.22).

```
F = zpk([], [-1 -2 -2], 2)
```

The $u(t)$ in Eq. (2.20) shows that the response is zero until $t = 0$. Unless otherwise specified, all inputs to systems in the text will not start until $t = 0$. Thus, output responses will also be zero until $t = 0$. For convenience, we will leave off the $u(t)$ notation from now on. Accordingly, we write the output response as

$$y(t) = 1 - 2e^{-4t} + e^{-8t} \quad (2.21)$$

Case 2. Roots of the Denominator of $F(s)$ Are Real and Repeated An example of an $F(s)$ with real and repeated roots in the denominator is

$$F(s) = \frac{2}{(s+1)(s+2)^2} \quad (2.22)$$

The roots of $(s+2)^2$ in the denominator are repeated, since the factor is raised to an integer power higher than 1. In this case, the denominator root at -2 is a *multiple root of multiplicity 2*.

We can write the partial-fraction expansion as a sum of terms, where each factor of the denominator forms the denominator of each term. In addition, each multiple root generates additional terms consisting of denominator factors of reduced multiplicity. For example, if

$$F(s) = \frac{2}{(s+1)(s+2)^2} = \frac{K_1}{s+1} + \frac{K_2}{(s+2)^2} + \frac{K_3}{s+2} \quad (2.23)$$

then $K_1 = 2$, which can be found as previously described. K_2 can be isolated by multiplying Eq. (2.23) by $(s+2)^2$, yielding

$$\frac{2}{s+1} = (s+2)^2 \frac{K_1}{s+1} + K_2 + (s+2)K_3 \quad (2.24)$$

Letting s approach -2 , $K_2 = -2$. To find K_3 we see that if we differentiate Eq. (2.24) with respect to s ,

$$\frac{-2}{(s+1)^2} = \frac{(s+2)s}{(s+1)^2} K_1 + K_3 \quad (2.25)$$

K_3 is isolated and can be found if we let s approach -2 . Hence, $K_3 = -2$.

Each component part of Eq. (2.23) is an $F(s)$ in Table 2.1; hence, $f(t)$ is the sum of the inverse Laplace transform of each term, or

$$f(t) = 2e^{-t} - 2te^{-2t} - 2e^{-2t} \quad (2.26)$$

If the denominator root is of higher multiplicity than 2, successive differentiation would isolate each residue in the expansion of the multiple root.

TryIt 2.2

Use the following MATLAB statements to help you get Eq. (2.26).

```
numf=2;
denf=poly([-1 -2 -2]);
[k,p,k]=residue...
(numf,denf)
```

In general, then, given an $F(s)$ whose denominator has real and repeated roots, a partial-fraction expansion,

$$\begin{aligned}
 F(s) &= \frac{N(s)}{D(s)} \\
 &= \frac{N(s)}{(s+p_1)^r (s+p_2) \cdots (s+p_n)} \\
 &= \frac{K_1}{(s+p_1)^r} + \frac{K_2}{(s+p_1)^{r-1}} + \cdots + \frac{K_r}{(s+p_1)} \\
 &\quad + \frac{K_{r+1}}{(s+p_2)} + \cdots + \frac{K_n}{(s+p_n)}
 \end{aligned} \tag{2.27}$$

can be made if the order of $N(s)$ is less than the order of $D(s)$ and the repeated roots are of multiplicity r at $-p_1$. To find K_1 through K_r for the roots of multiplicity greater than unity, first multiply Eq. (2.27) by $(s+p_1)^r$ getting $F_1(s)$, which is

$$\begin{aligned}
 F_1(s) &= (s+p_1)^r F(s) \\
 &= \frac{(s+p_1)^r N(s)}{(s+p_1)^r (s+p_2) \cdots (s+p_n)} \\
 &= K_1 + (s+p_1)K_2 + (s+p_1)^2 K_3 + \cdots + (s+p_1)^{r-1} K_r \\
 &\quad + \frac{K_{r+1}(s+p_1)^r}{(s+p_2)} + \cdots + \frac{K_n(s+p_1)^r}{(s+p_n)}
 \end{aligned} \tag{2.28}$$

Immediately, we can solve for K_1 if we let s approach $-p_1$. We can solve for K_2 if we differentiate Eq. (2.28) with respect to s and then let s approach $-p_1$. Subsequent differentiation will allow us to find K_3 through K_r . The general expression for K_1 through K_r for the multiple roots is

$$K_i = \frac{1}{(i-1)!} \left. \frac{d^{i-1} F_1(s)}{ds^{i-1}} \right|_{s \rightarrow -p_1} \quad i = 1, 2, \dots, r; \quad 0! = 1 \tag{2.29}$$

Case 3. Roots of the Denominator of $F(s)$ Are Complex or Imaginary An example of $F(s)$ with complex roots in the denominator is

$$F(s) = \frac{3}{s(s^2 + 2s + 5)} \tag{2.30}$$

This function can be expanded in the following form:

$$\frac{3}{s(s^2 + 2s + 5)} = \frac{K_1}{s} + \frac{K_2 s + K_3}{s^2 + 2s + 5} \tag{2.31}$$

K_1 is found in the usual way to be $\frac{3}{5}$. K_2 and K_3 can be found by first multiplying Eq. (2.31) by the lowest common denominator, $s(s^2 + 2s + 5)$, and clearing the fractions. After simplification with $K_1 = \frac{3}{5}$, we obtain

$$3 = \left(K_2 + \frac{3}{5} \right) s^2 + \left(K_3 + \frac{6}{5} \right) s + 3 \tag{2.32}$$

TryIt 2.3

Use the following MATLAB and Control System Toolbox statement to form the LTI transfer function of Eq. (2.30).

```
F = tf([3],[1 2 5 0])
```

Balancing coefficients, $(K_2 + \frac{3}{5}) = 0$ and $(K_3 + \frac{6}{5}) = 0$. Hence $K_2 = -\frac{3}{5}$ and $K_3 = -\frac{6}{5}$. Thus,

$$F(s) = \frac{3}{s(s^2 + 2s + 5)} = \frac{3/5}{s} - \frac{3}{5} \frac{s+2}{s^2 + 2s + 5} \quad (2.33)$$

The last term can be shown to be the sum of the Laplace transforms of an exponentially damped sine and cosine. Using Item 7 in Table 2.1 and Items 2 and 4 in Table 2.2, we get

$$\mathcal{L}[Ae^{-at} \cos \omega t] = \frac{A(s+a)}{(s+a)^2 + \omega^2} \quad (2.34)$$

Similarly,

$$\mathcal{L}[Be^{-at} \sin \omega t] = \frac{B\omega}{(s+a)^2 + \omega^2} \quad (2.35)$$

Adding Eqs. (2.34) and (2.35), we get

$$\mathcal{L}[Ae^{-at} \cos \omega t + Be^{-at} \sin \omega t] = \frac{A(s+a) + B\omega}{(s+a)^2 + \omega^2} \quad (2.36)$$

We now convert the last term of Eq. (2.33) to the form suggested by Eq. (2.36) by completing the squares in the denominator and adjusting terms in the numerator without changing its value. Hence

$$F(s) = \frac{3/5}{s} - \frac{3(s+1) + (1/2)(2)}{5(s+1)^2 + 2^2} \quad (2.37)$$

Comparing Eq. (2.37) to Table 2.1 and Eq. (2.36), we find

$$f(t) = \frac{3}{5} - \frac{3}{5} e^{-t} \left(\cos 2t + \frac{1}{2} \sin 2t \right) \quad (2.38)$$

In order to visualize the solution, an alternate form of $f(t)$, obtained by trigonometric identities, is preferable. Using the amplitudes of the cos and sin terms, we factor out $\sqrt{1^2 + (1/2)^2}$ from the term in parentheses and obtain

$$f(t) = \frac{3}{5} - \frac{3}{5} \sqrt{1^2 + (1/2)^2} e^{-t} \left(\frac{1}{\sqrt{1^2 + (1/2)^2}} \cos 2t + \frac{1/2}{\sqrt{1^2 + (1/2)^2}} \sin 2t \right) \quad (2.39)$$

Letting $1/\sqrt{1^2 + (1/2)^2} = \cos \phi$ and $(1/2)/\sqrt{1^2 + (1/2)^2} = \sin \phi$,

$$f(t) = \frac{3}{5} - \frac{3}{5} \sqrt{1^2 + (1/2)^2} e^{-t} (\cos \phi \cos 2t + \sin \phi \sin 2t) \quad (2.40)$$

or

$$f(t) = 0.6 - 0.671e^{-t} \cos(2t - \phi) \quad (2.41)$$

TryIt 2.4

Use the following MATLAB and Symbolic Math Toolbox statements to get Eq. (2.38) from Eq. (2.30).

```
syms s
f=ilaplace...
(3/(s*(s^2+2*s+5)));
pretty(f)
```

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where $\phi = \arctan 0.5 = 26.57^\circ$. Thus, $f(t)$ is a constant plus an exponentially damped sinusoid.

In general, then, given an $F(s)$ whose denominator has complex or purely imaginary roots, a partial-fraction expansion,

$$\begin{aligned} F(s) &= \frac{N(s)}{D(s)} = \frac{N(s)}{(s+p_1)(s^2+as+b)\cdots} \\ &= \frac{K_1}{(s+p_1)} + \frac{(K_2s+K_3)}{(s^2+as+b)} + \cdots \end{aligned} \quad (2.42)$$

can be made if the order of $N(s)$ is less than the order of $D(s)$ p_1 is real, and (s^2+as+b) has complex or purely imaginary roots. The complex or imaginary roots are expanded with (K_2s+K_3) terms in the numerator rather than just simply K_i , as in the case of real roots. The K_i 's in Eq. (2.42) are found through balancing the coefficients of the equation after clearing fractions. After completing the squares on (s^2+as+b) and adjusting the numerator, $(K_2s+K_3)/(s^2+as+b)$ can be put into the form shown on the right-hand side of Eq. (2.36).

Finally, the case of purely imaginary roots arises if $a=0$ in Eq. (2.42). The calculations are the same.

Another method that follows the technique used for the partial-fraction expansion of $F(s)$ with real roots in the denominator can be used for complex and imaginary roots. However, the residues of the complex and imaginary roots are themselves complex conjugates. Then, after taking the inverse Laplace transform, the resulting terms can be identified as

$$\frac{e^{j\theta} + e^{-j\theta}}{2} = \cos \theta \quad (2.43)$$

and

$$\frac{e^{j\theta} - e^{-j\theta}}{2j} = \sin \theta \quad (2.44)$$

For example, the previous $F(s)$ can also be expanded in partial fractions as

$$\begin{aligned} F(s) &= \frac{3}{s(s^2+2s+5)} = \frac{3}{s(s+1+j2)(s+1-j2)} \\ &= \frac{K_1}{s} + \frac{K_2}{s+1+j2} + \frac{K_3}{s+1-j2} \end{aligned} \quad (2.45)$$

Finding K_2 ,

$$K_2 = \frac{3}{s(s+1-j2)} \Big|_{s \rightarrow -1-j2} = -\frac{3}{20}(2+j1) \quad (2.46)$$

Similarly, K_3 is found to be the complex conjugate of K_2 , and K_1 is found as previously described. Hence,

$$F(s) = \frac{3/5}{s} - \frac{3}{20} \left(\frac{2+j1}{s+1+j2} + \frac{2-j1}{s+1-j2} \right) \quad (2.47)$$

from which

$$\begin{aligned} f(t) &= \frac{3}{5} - \frac{3}{20} \left[(2+j1)e^{-(1+j2)t} + (2-j1)e^{-(1-j2)t} \right] \\ &= \frac{3}{5} - \frac{3}{20} e^{-t} \left[4 \left(\frac{e^{j2t} + e^{-j2t}}{2} \right) + 2 \left(\frac{e^{j2t} - e^{-j2t}}{2j} \right) \right] \end{aligned} \quad (2.48)$$

TryIt 2.5

Use the following MATLAB statements to help you get Eq. (2.47).

```
numf=3
denf=[1 2 5 0]
[k,p,k]=residue...
    (numf,denf)
```

Using Eqs. (2.43) and (2.44), we get

$$f(t) = \frac{3}{5} - \frac{3}{5}e^{-t} \left(\cos 2t + \frac{1}{2} \sin 2t \right) = 0.6 - 0.671e^{-t} \cos(2t - \phi) \quad (2.49)$$

where $\phi = \arctan 0.5 = 26.57^\circ$.

Symbolic Math

SM

Students who are performing the MATLAB exercises and want to explore the added capability of MATLAB's Symbolic Math Toolbox should now run `ch2sp1` and `ch2sp2` in Appendix F at www.wiley.com/college/nise. You will learn how to construct symbolic objects and then find the inverse Laplace and Laplace transforms of frequency and time functions, respectively. The examples in Case 2 and Case 3 in this section will be solved using the Symbolic Math Toolbox.

Skill-Assessment Exercise 2.1

PROBLEM: Find the Laplace transform of $f(t) = te^{-5t}$.

ANSWER: $F(s) = 1/(s + 5)^2$

The complete solution is at www.wiley.com/college/nise.

Skill-Assessment Exercise 2.2

PROBLEM: Find the inverse Laplace transform of $F(s) = 10/[s(s + 2)(s + 3)^2]$.

ANSWER: $f(t) = \frac{5}{9} - 5e^{-2t} + \frac{10}{3}te^{-3t} + \frac{40}{9}e^{-3t}$

The complete solution is at www.wiley.com/college/nise.

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Control Solutions

2.3 The Transfer Function

In the previous section we defined the Laplace transform and its inverse. We presented the idea of the partial-fraction expansion and applied the concepts to the solution of differential equations. We are now ready to formulate the system representation shown in Figure 2.1 by establishing a viable definition for a function that algebraically relates a system's output to its input. This function will allow separation of the input, system, and output into three separate and distinct parts, unlike the differential equation. The function will also allow us to *algebraically* combine mathematical representations of subsystems to yield a total system representation.

Let us begin by writing a general n th-order, linear, time-invariant differential equation,

$$a_n \frac{d^n c(t)}{dt^n} + a_{n-1} \frac{d^{n-1} c(t)}{dt^{n-1}} + \cdots + a_0 c(t) = b_m \frac{d^m r(t)}{dt^m} + b_{m-1} \frac{d^{m-1} r(t)}{dt^{m-1}} + \cdots + b_0 r(t) \quad (2.50)$$