

and entering MATLAB Program 7-4 into the computer, we obtain the Bode diagram shown in Figure 7-24.

MATLAB Program 7-4
A = [0 1;-25 -4];
B = [0;25];
C = [1 0];
D = [0];
bode(A,B,C,D)
title('Bode Diagram')

If we replace the command `bode(A,B,C,D)` in MATLAB Program 7-4 with

```
bode(A,B,C,D,1)
```

then MATLAB will produce the Bode diagram identical to that shown in Figure 7-24.

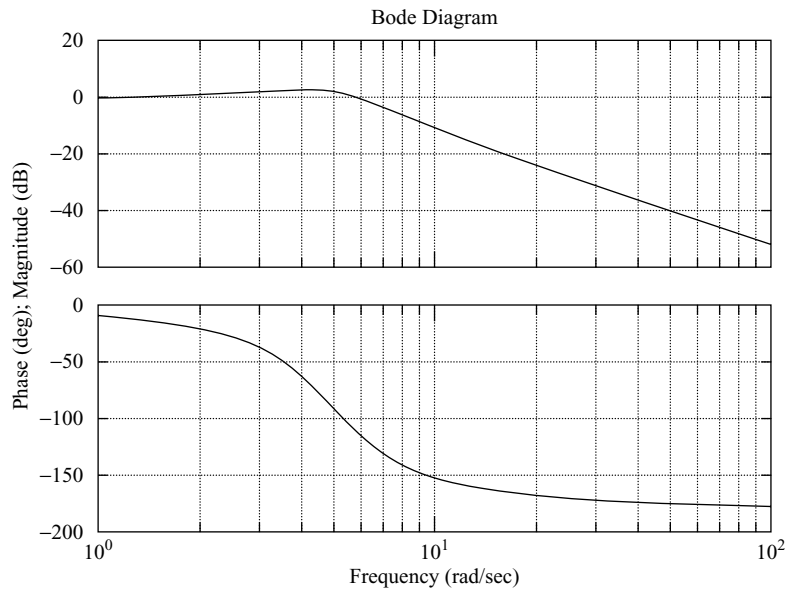


Figure 7-24
Bode diagram of the system considered in Example 7-7.

7-3 POLAR PLOTS

The polar plot of a sinusoidal transfer function $G(j\omega)$ is a plot of the magnitude of $G(j\omega)$ versus the phase angle of $G(j\omega)$ on polar coordinates as ω is varied from zero to infinity. Thus, the polar plot is the locus of vectors $|G(j\omega)| \angle G(j\omega)$ as ω is varied from zero to infinity. Note that in polar plots a positive (negative) phase angle is measured counterclockwise (clockwise) from the positive real axis. The polar plot is often called the Nyquist plot. An example of such a plot is shown in Figure 7-25. Each point on the polar plot of $G(j\omega)$ represents the terminal point of a vector at a particular value of ω . In the polar plot, it is important to show the frequency graduation of the locus. The projections of $G(j\omega)$ on the real and imaginary axes are its real and imaginary components.

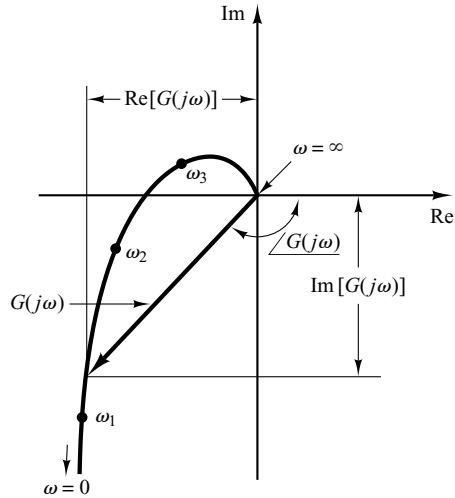


Figure 7-25
Polar plot.

MATLAB may be used to obtain a polar plot $G(j\omega)$ or to obtain $|G(j\omega)|$ and $\angle G(j\omega)$ accurately for various values of ω in the frequency range of interest.

An advantage in using a polar plot is that it depicts the frequency-response characteristics of a system over the entire frequency range in a single plot. One disadvantage is that the plot does not clearly indicate the contributions of each individual factor of the open-loop transfer function.

Integral and Derivative Factors ($j\omega$) $^{\mp 1}$. The polar plot of $G(j\omega) = 1/j\omega$ is the negative imaginary axis, since

$$G(j\omega) = \frac{1}{j\omega} = -j \frac{1}{\omega} = \frac{1}{\omega} \angle -90^\circ$$

The polar plot of $G(j\omega) = j\omega$ is the positive imaginary axis.

First-Order Factors ($1 + j\omega T$) $^{\mp 1}$. For the sinusoidal transfer function

$$G(j\omega) = \frac{1}{1 + j\omega T} = \frac{1}{\sqrt{1 + \omega^2 T^2}} \angle -\tan^{-1} \omega T$$

the values of $G(j\omega)$ at $\omega = 0$ and $\omega = 1/T$ are, respectively,

$$G(j0) = 1 \angle 0^\circ \quad \text{and} \quad G\left(j \frac{1}{T}\right) = \frac{1}{\sqrt{2}} \angle -45^\circ$$

If ω approaches infinity, the magnitude of $G(j\omega)$ approaches zero and the phase angle approaches -90° . The polar plot of this transfer function is a semicircle as the frequency ω is varied from zero to infinity, as shown in Figure 7-26(a). The center is located at 0.5 on the real axis, and the radius is equal to 0.5.

To prove that the polar plot of the first-order factor $G(j\omega) = 1/(1 + j\omega T)$ is a semicircle, define

$$G(j\omega) = X + jY$$

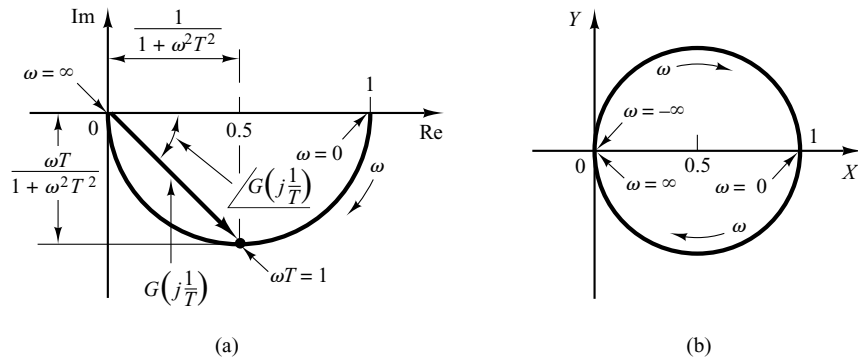


Figure 7-26
 (a) Polar plot of $1/(1 + j\omega T)$; (b) plot of $G(j\omega)$ in X - Y plane.

where

$$X = \frac{1}{1 + \omega^2 T^2} = \text{real part of } G(j\omega)$$

$$Y = \frac{-\omega T}{1 + \omega^2 T^2} = \text{imaginary part of } G(j\omega)$$

Then we obtain

$$\left(X - \frac{1}{2}\right)^2 + Y^2 = \left(\frac{1}{2} \frac{1 - \omega^2 T^2}{1 + \omega^2 T^2}\right)^2 + \left(\frac{-\omega T}{1 + \omega^2 T^2}\right)^2 = \left(\frac{1}{2}\right)^2$$

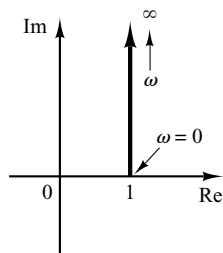


Figure 7-27
 Polar plot of $1 + j\omega T$.

Thus, in the X - Y plane $G(j\omega)$ is a circle with center at $X = \frac{1}{2}$, $Y = 0$ and with radius $\frac{1}{2}$, as shown in Figure 7-26(b). The lower semicircle corresponds to $0 \leq \omega \leq \infty$, and the upper semicircle corresponds to $-\infty \leq \omega \leq 0$.

The polar plot of the transfer function $1 + j\omega T$ is simply the upper half of the straight line passing through point $(1, 0)$ in the complex plane and parallel to the imaginary axis, as shown in Figure 7-27. The polar plot of $1 + j\omega T$ has an appearance completely different from that of $1/(1 + j\omega T)$.

Quadratic Factors $[1 + 2\zeta(j\omega/\omega_n) + (j\omega/\omega_n)^2]^{-1}$. The low- and high-frequency portions of the polar plot of the following sinusoidal transfer function

$$G(j\omega) = \frac{1}{1 + 2\zeta\left(j\frac{\omega}{\omega_n}\right) + \left(j\frac{\omega}{\omega_n}\right)^2}, \quad \text{for } \zeta > 0$$

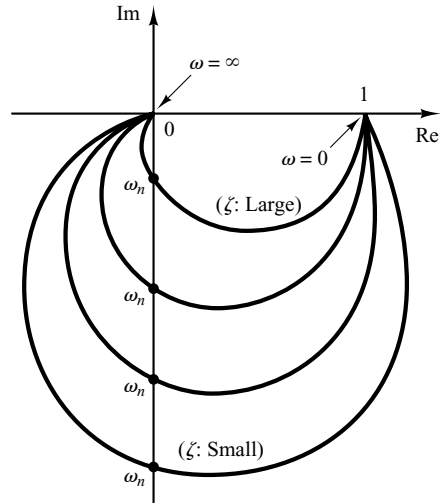
are given, respectively, by

$$\lim_{\omega \rightarrow 0} G(j\omega) = 1 \angle 0^\circ \quad \text{and} \quad \lim_{\omega \rightarrow \infty} G(j\omega) = 0 \angle -180^\circ$$

The polar plot of this sinusoidal transfer function starts at $1 \angle 0^\circ$ and ends at $0 \angle -180^\circ$ as ω increases from zero to infinity. Thus, the high-frequency portion of $G(j\omega)$ is tangent to the negative real axis.

Figure 7-28
Polar plots of

$$\frac{1}{1 + 2\zeta\left(j\frac{\omega}{\omega_n}\right) + \left(j\frac{\omega}{\omega_n}\right)^2} \text{ for } \zeta > 0.$$



Examples of polar plots of the transfer function just considered are shown in Figure 7-28. The exact shape of a polar plot depends on the value of the damping ratio ζ , but the general shape of the plot is the same for both the underdamped case ($1 > \zeta > 0$) and overdamped case ($\zeta > 1$).

For the underdamped case at $\omega = \omega_n$, we have $G(j\omega_n) = 1/(j2\zeta)$, and the phase angle at $\omega = \omega_n$ is -90° . Therefore, it can be seen that the frequency at which the $G(j\omega)$ locus intersects the imaginary axis is the undamped natural frequency ω_n . In the polar plot, the frequency point whose distance from the origin is maximum corresponds to the resonant frequency ω_r . The peak value of $G(j\omega)$ is obtained as the ratio of the magnitude of the vector at the resonant frequency ω_r to the magnitude of the vector at $\omega = 0$. The resonant frequency ω_r is indicated in the polar plot shown in Figure 7-29.

For the overdamped case, as ζ increases well beyond unity, the $G(j\omega)$ locus approaches a semicircle. This may be seen from the fact that, for a heavily damped system, the characteristic roots are real, and one is much smaller than the other. Since, for sufficiently large ζ , the effect of the larger root (larger in the absolute value) on the response becomes very small, the system behaves like a first-order one.

Figure 7-29
Polar plot showing the resonant peak and resonant frequency ω_r .

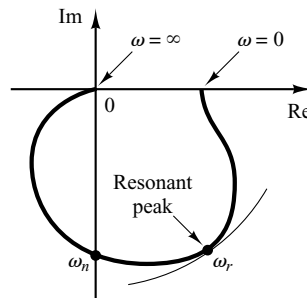
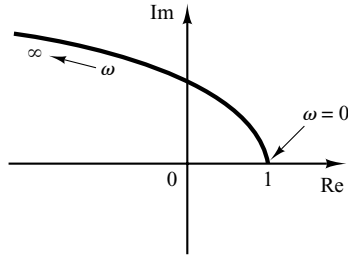


Figure 7-30

Polar plot of

$$1 + 2\zeta\left(j\frac{\omega}{\omega_n}\right) + \left(j\frac{\omega}{\omega_n}\right)^2 \text{ for } \zeta > 0.$$



Next, consider the following sinusoidal transfer function:

$$\begin{aligned} G(j\omega) &= 1 + 2\zeta\left(j\frac{\omega}{\omega_n}\right) + \left(j\frac{\omega}{\omega_n}\right)^2 \\ &= \left(1 - \frac{\omega^2}{\omega_n^2}\right) + j\left(\frac{2\zeta\omega}{\omega_n}\right) \end{aligned}$$

The low-frequency portion of the curve is

$$\lim_{\omega \rightarrow 0} G(j\omega) = 1 \angle 0^\circ$$

and the high-frequency portion is

$$\lim_{\omega \rightarrow \infty} G(j\omega) = \infty \angle 180^\circ$$

Since the imaginary part of $G(j\omega)$ is positive for $\omega > 0$ and is monotonically increasing, and the real part of $G(j\omega)$ is monotonically decreasing from unity, the general shape of the polar plot of $G(j\omega)$ is as shown in Figure 7-30. The phase angle is between 0° and 180° .

EXAMPLE 7-8 Consider the following second-order transfer function:

$$G(s) = \frac{1}{s(Ts + 1)}$$

Sketch a polar plot of this transfer function.

Since the sinusoidal transfer function can be written

$$G(j\omega) = \frac{1}{j\omega(1 + j\omega T)} = -\frac{T}{1 + \omega^2 T^2} - j\frac{1}{\omega(1 + \omega^2 T^2)}$$

the low-frequency portion of the polar plot becomes

$$\lim_{\omega \rightarrow 0} G(j\omega) = -T - j\infty$$

and the high-frequency portion becomes

$$\lim_{\omega \rightarrow \infty} G(j\omega) = 0 - j0$$

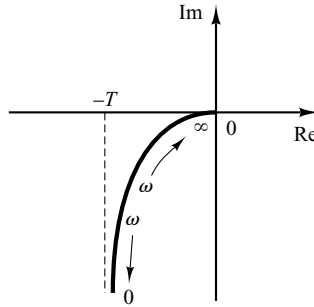


Figure 7-31
Polar plot of
 $1/[j\omega(1 + j\omega T)]$.

The general shape of the polar plot of $G(j\omega)$ is shown in Figure 7-31. The $G(j\omega)$ plot is asymptotic to the vertical line passing through the point $(-T, 0)$. Since this transfer function involves an integrator ($1/s$), the general shape of the polar plot differs substantially from those of second-order transfer functions that do not have an integrator.

EXAMPLE 7-9 Obtain the polar plot of the following transfer function:

$$G(j\omega) = \frac{e^{-j\omega L}}{1 + j\omega T}$$

Since $G(j\omega)$ can be written

$$G(j\omega) = (e^{-j\omega L}) \left(\frac{1}{1 + j\omega T} \right)$$

the magnitude and phase angle are, respectively,

$$|G(j\omega)| = |e^{-j\omega L}| \cdot \left| \frac{1}{1 + j\omega T} \right| = \frac{1}{\sqrt{1 + \omega^2 T^2}}$$

and

$$\angle G(j\omega) = \angle e^{-j\omega L} + \angle \frac{1}{1 + j\omega T} = -\omega L - \tan^{-1} \omega T$$

Since the magnitude decreases from unity monotonically and the phase angle also decreases monotonically and indefinitely, the polar plot of the given transfer function is a spiral, as shown in Figure 7-32.

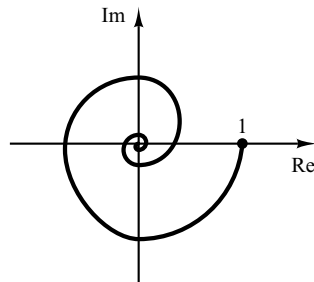


Figure 7-32
Polar plot of
 $e^{-j\omega L}/(1 + j\omega T)$.

General Shapes of Polar Plots. The polar plots of a transfer function of the form

$$G(j\omega) = \frac{K(1 + j\omega T_a)(1 + j\omega T_b) \cdots}{(j\omega)^\lambda (1 + j\omega T_1)(1 + j\omega T_2) \cdots}$$

$$= \frac{b_0(j\omega)^m + b_1(j\omega)^{m-1} + \cdots}{a_0(j\omega)^n + a_1(j\omega)^{n-1} + \cdots}$$

where $n > m$ or the degree of the denominator polynomial is greater than that of the numerator, will have the following general shapes:

1. For $\lambda = 0$ or type 0 systems: The starting point of the polar plot (which corresponds to $\omega = 0$) is finite and is on the positive real axis. The tangent to the polar plot at $\omega = 0$ is perpendicular to the real axis. The terminal point, which corresponds to $\omega = \infty$, is at the origin, and the curve is tangent to one of the axes.
2. For $\lambda = 1$ or type 1 systems: the $j\omega$ term in the denominator contributes -90° to the total phase angle of $G(j\omega)$ for $0 \leq \omega \leq \infty$. At $\omega = 0$, the magnitude of $G(j\omega)$ is infinity, and the phase angle becomes -90° . At low frequencies, the polar plot is asymptotic to a line parallel to the negative imaginary axis. At $\omega = \infty$, the magnitude becomes zero, and the curve converges to the origin and is tangent to one of the axes.
3. For $\lambda = 2$ or type 2 systems: The $(j\omega)^2$ term in the denominator contributes -180° to the total phase angle of $G(j\omega)$ for $0 \leq \omega \leq \infty$. At $\omega = 0$, the magnitude of $G(j\omega)$ is infinity, and the phase angle is equal to -180° . At low frequencies, the polar plot may be asymptotic to the negative real axis. At $\omega = \infty$, the magnitude becomes zero, and the curve is tangent to one of the axes.

The general shapes of the low-frequency portions of the polar plots of type 0, type 1, and type 2 systems are shown in Figure 7–33. It can be seen that, if the degree of the

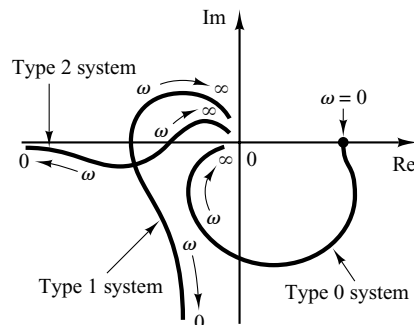


Figure 7–33
Polar plots of type 0, type 1, and type 2 systems.

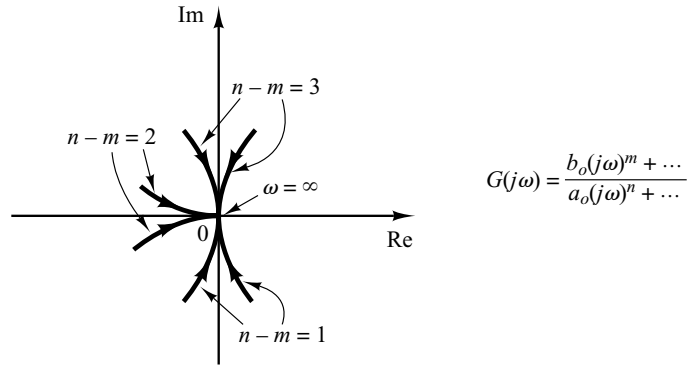


Figure 7-34
Polar plots in the high-frequency range.

denominator polynomial of $G(j\omega)$ is greater than that of the numerator, then the $G(j\omega)$ loci converge to the origin clockwise. At $\omega = \infty$, the loci are tangent to one or the other axes, as shown in Figure 7-34.

Note that any complicated shapes in the polar plot curves are caused by the numerator dynamics—that is, by the time constants in the numerator of the transfer function. Figure 7-35 shows examples of polar plots of transfer functions with numerator dynamics. In analyzing control systems, the polar plot of $G(j\omega)$ in the frequency range of interest must be accurately determined.

Table 7-1 shows sketches of polar plots of several transfer functions.

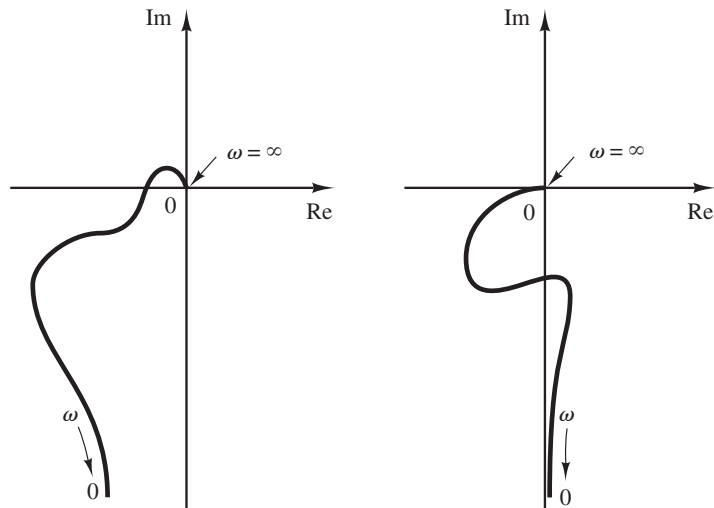


Figure 7-35
Polar plots of transfer functions with numerator dynamics.

Table 7-1 Polar Plots of Simple Transfer Functions

<p>Polar plot of $\frac{1}{j\omega}$. The plot shows a vertical line on the negative imaginary axis. The point at $\omega = 0$ is at the origin (0,0), and the point at $\omega = \infty$ is at the top of the axis. The real axis is labeled 'Re' and the imaginary axis is labeled 'Im'.</p>	<p>Polar plot of $\frac{1+j\omega T}{j\omega T}$. The plot shows a vertical line on the positive imaginary axis. The point at $\omega = 0$ is at the origin (0,0), and the point at $\omega = \infty$ is at the top of the axis. The real axis is labeled 'Re' and the imaginary axis is labeled 'Im'.</p>
<p>Polar plot of $j\omega$. The plot shows a vertical line on the positive imaginary axis. The point at $\omega = 0$ is at the origin (0,0), and the point at $\omega = \infty$ is at the top of the axis. The real axis is labeled 'Re' and the imaginary axis is labeled 'Im'.</p>	<p>Polar plot of $1+j\omega T$. The plot shows a vertical line on the positive imaginary axis. The point at $\omega = 0$ is at the origin (0,0), and the point at $\omega = \infty$ is at the top of the axis. The real axis is labeled 'Re' and the imaginary axis is labeled 'Im'.</p>
<p>Polar plot of $\frac{1}{(j\omega)^2}$. The plot shows a horizontal line on the negative real axis. The point at $\omega = 0$ is at the origin (0,0), and the point at $\omega = \infty$ is at the left of the axis. The real axis is labeled 'Re' and the imaginary axis is labeled 'Im'.</p>	<p>Polar plot of $\frac{j\omega T}{1+j\omega T}$. The plot shows a semicircle in the first quadrant. The point at $\omega = 0$ is at the origin (0,0), and the point at $\omega = \infty$ is at (1,0) on the real axis. The real axis is labeled 'Re' and the imaginary axis is labeled 'Im'.</p>
<p>Polar plot of $\frac{1}{1+j\omega a}$ ($a > 1$). The plot shows a semicircle in the fourth quadrant. The point at $\omega = 0$ is at (1,0) on the real axis, and the point at $\omega = \infty$ is at the origin (0,0). The real axis is labeled 'Re' and the imaginary axis is labeled 'Im'.</p>	<p>Polar plot of $\frac{1}{(1+j\omega T_1)(1+j\omega T_2)(1+j\omega T_3)}$. The plot shows a curve in the fourth quadrant. The point at $\omega = 0$ is at (1,0) on the real axis, and the point at $\omega = \infty$ is at the origin (0,0). The real axis is labeled 'Re' and the imaginary axis is labeled 'Im'.</p>
<p>Polar plot of $\frac{\omega_n^2}{j\omega[(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2]}$. The plot shows a curve in the fourth quadrant. The point at $\omega = 0$ is at the origin (0,0), and the point at $\omega = \infty$ is at the origin (0,0). The real axis is labeled 'Re' and the imaginary axis is labeled 'Im'.</p>	<p>Polar plot of $\frac{1+j\omega T_1}{j\omega(1+j\omega T_2)(1+j\omega T_3)}$. The plot shows a curve in the fourth quadrant. The point at $\omega = 0$ is at the origin (0,0), and the point at $\omega = \infty$ is at the origin (0,0). The real axis is labeled 'Re' and the imaginary axis is labeled 'Im'.</p>

Drawing Nyquist Plots with MATLAB. Nyquist plots, just like Bode diagrams, are commonly used in the frequency-response representation of linear, time-invariant, feedback control systems. Nyquist plots are polar plots, while Bode diagrams are rectangular plots. One plot or the other may be more convenient for a particular operation, but a given operation can always be carried out in either plot.

The MATLAB command `nyquist` computes the frequency response for continuous-time, linear, time-invariant systems. When invoked without left-hand arguments, `nyquist` produces a Nyquist plot on the screen.

The command

```
nyquist(num,den)
```

draws the Nyquist plot of the transfer function

$$G(s) = \frac{\text{num}(s)}{\text{den}(s)}$$

where num and den contain the polynomial coefficients in descending powers of s . Other commonly used nyquist commands are

```
nyquist(num,den,w)
nyquist(A,B,C,D)
nyquist(A,B,C,D,w)
nyquist(A,B,C,D,iu,w)
nyquist(sys)
```

The command involving the user-specified frequency vector w , such as

```
nyquist(num,den,w)
```

calculates the frequency response at the specified frequency points in radians per second.

When invoked with left-hand arguments such as

```
[re,im,w] = nyquist(num,den)
[re,im,w] = nyquist(num,den,w)
[re,im,w] = nyquist(A,B,C,D)
[re,im,w] = nyquist(A,B,C,D,w)
[re,im,w] = nyquist(A,B,C,D,iu,w)
[re,im,w] = nyquist(sys)
```

MATLAB returns the frequency response of the system in the matrices re , im , and w . No plot is drawn on the screen. The matrices re and im contain the real and imaginary parts of the frequency response of the system, evaluated at the frequency points specified in the vector w . Note that re and im have as many columns as outputs and one row for each element in w .

EXAMPLE 7–10 Consider the following open-loop transfer function:

$$G(s) = \frac{1}{s^2 + 0.8s + 1}$$

Draw a Nyquist plot with MATLAB.

Since the system is given in the form of the transfer function, the command

```
nyquist(num,den)
```

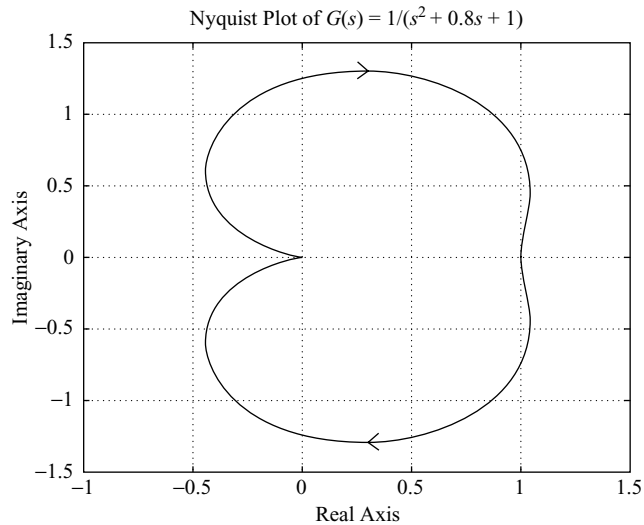
may be used to draw a Nyquist plot. MATLAB Program 7–5 produces the Nyquist plot shown in Figure 7–36. In this plot, the ranges for the real axis and imaginary axis are automatically determined.

MATLAB Program 7-5

```

num = [1];
den = [1 0.8 1];
nyquist(num,den)
grid
title('Nyquist Plot of G(s) = 1/(s^2 + 0.8s + 1)')

```

**Figure 7-36**

Nyquist plot of

$$G(s) = \frac{1}{s^2 + 0.8s + 1}.$$

If we wish to draw the Nyquist plot using manually determined ranges—for example, from -2 to 2 on the real axis and from -2 to 2 on the imaginary axis—enter the following command into the computer:

```

v = [-2 2 -2 2];
axis(v);

```

or, combining these two lines into one,

```

axis([-2 2 -2 2]);

```

See MATLAB Program 7-6 and the resulting Nyquist plot shown in Figure 7-37.

MATLAB Program 7-6

```

% ----- Nyquist plot -----
num = [1];
den = [1 0.8 1];
nyquist(num,den)
v = [-2 2 -2 2]; axis(v)
grid
title('Nyquist Plot of G(s) = 1/(s^2 + 0.8s + 1)')

```

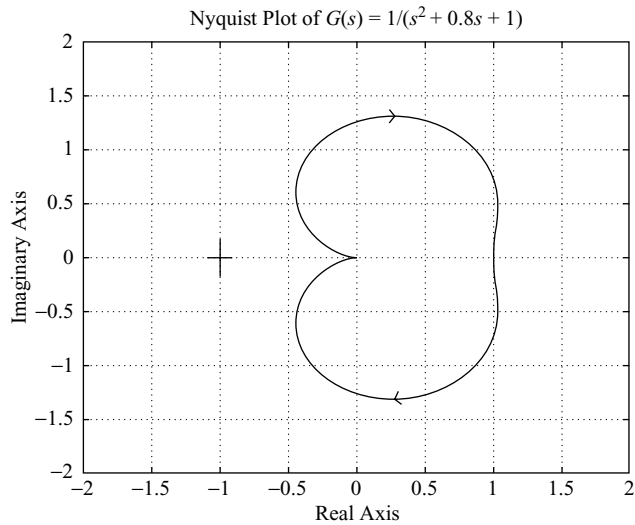


Figure 7-37
Nyquist plot of
 $G(s) = \frac{1}{s^2 + 0.8s + 1}$.

Caution. In drawing a Nyquist plot, where a MATLAB operation involves “Divide by zero,” the resulting Nyquist plot may have an erroneous or undesirable appearance. For example, if the transfer function $G(s)$ is given by

$$G(s) = \frac{1}{s(s + 1)}$$

then the MATLAB command

```
num = [1];
den = [1 1 0];
nyquist(num,den)
```

produces an undesirable Nyquist plot. An example of an undesirable Nyquist plot is shown in Figure 7-38. If such an undesirable Nyquist plot appears on the computer,

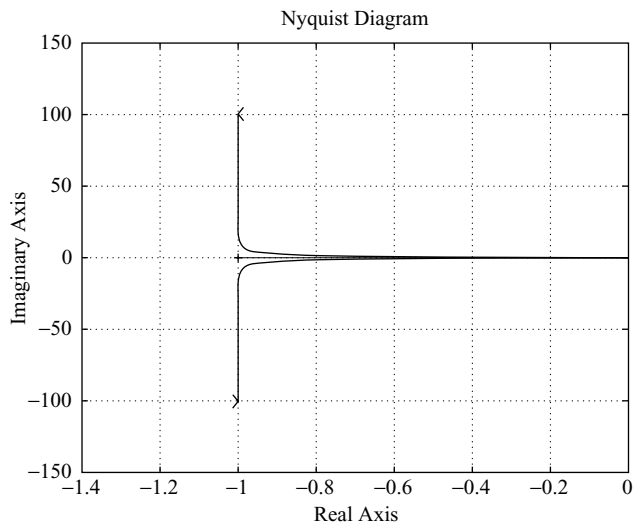


Figure 7-38
Undesirable Nyquist plot.

then it can be corrected if we specify the axis(v). For example, if we enter the axis command

$$v = [-2 \ 2 \ -5 \ 5]; \text{axis}(v)$$

in the computer, then a desirable form of Nyquist plot can be obtained. See Example 7–11.

EXAMPLE 7–11 Draw a Nyquist plot for the following $G(s)$:

$$G(s) = \frac{1}{s(s+1)}$$

MATLAB Program 7–7 will produce a desirable form of Nyquist plot on the computer, even though a warning message “Divide by zero” may appear on the screen. The resulting Nyquist plot is shown in Figure 7–39.

```

MATLAB Program 7–7
% ----- Nyquist plot-----
num = [1];
den = [1 1 0];
nyquist(num,den)
v = [-2 2 -5 5]; axis(v)
grid
title('Nyquist Plot of G(s) = 1/[s(s + 1)]')

```

Notice that the Nyquist plot shown in Figure 7–39 includes the loci for both $\omega > 0$ and $\omega < 0$. If we wish to draw the Nyquist plot for only the positive frequency region ($\omega > 0$), then we need to use the command

$$[re,im,w]=nyquist(num,den,w)$$

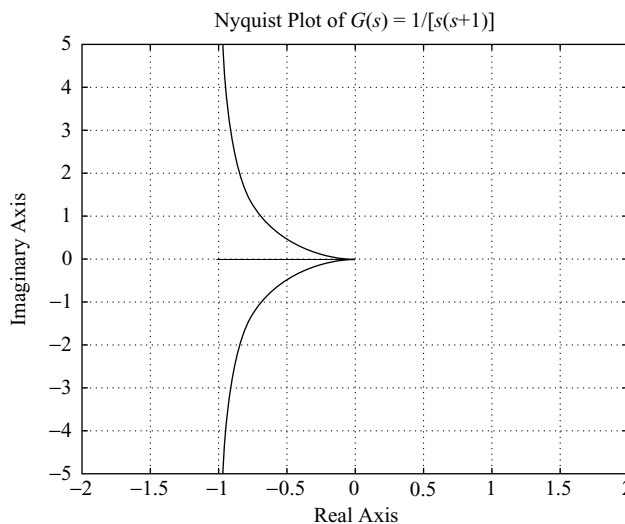


Figure 7–39
Nyquist plot of
 $G(s) = \frac{1}{s(s+1)}$.

A MATLAB program using this nyquist command is shown in MATLAB Program 7–8. The resulting Nyquist plot is presented in Figure 7–40.

MATLAB Program 7–8

```
% ----- Nyquist plot-----
num = [1];
den = [1 1 0];
w = 0.1:0.1:100;
[re,im,w] = nyquist(num,den,w);
plot(re,im)
v = [-2 2 -5 5]; axis(v)
grid
title('Nyquist Plot of G(s) = 1/[s(s + 1)]')
xlabel('Real Axis')
ylabel('Imag Axis')
```

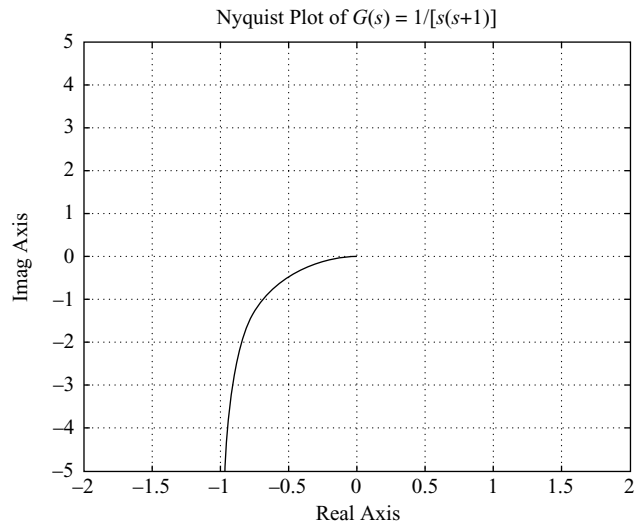


Figure 7–40
Nyquist plot of
 $G(s) = \frac{1}{s(s + 1)}$
for $\omega > 0$.

Drawing Nyquist Plots of a System Defined in State Space. Consider the system defined by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

where \mathbf{x} = state vector (n -vector)

\mathbf{y} = output vector (m -vector)

\mathbf{u} = control vector (r -vector)

\mathbf{A} = state matrix ($n \times n$ matrix)

\mathbf{B} = control matrix ($n \times r$ matrix)

\mathbf{C} = output matrix ($m \times n$ matrix)

\mathbf{D} = direct transmission matrix ($m \times r$ matrix)

Nyquist plots for this system may be obtained by the use of the command

```
nyquist(A,B,C,D)
```

This command produces a series of Nyquist plots, one for each input and output combination of the system. The frequency range is automatically determined.

The command

```
nyquist(A,B,C,D,iu)
```

produces Nyquist plots from the single input iu to all the outputs of the system, with the frequency range determined automatically. The scalar iu is an index into the inputs of the system and specifies which input to use for the frequency response.

The command

```
nyquist(A,B,C,D,iu,w)
```

uses the user-supplied frequency vector w . The vector w specifies the frequencies in radians per second at which the frequency response should be calculated.

EXAMPLE 7-12 Consider the system defined by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -25 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 25 \end{bmatrix} u$$
$$y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0] u$$

Draw a Nyquist plot.

This system has a single input u and a single output y . A Nyquist plot may be obtained by entering the command

```
nyquist(A,B,C,D)
```

or

```
nyquist(A,B,C,D,1)
```

MATLAB Program 7-9 will provide the Nyquist plot. (Note that we obtain the identical result by using either of these two commands.) Figure 7-41 shows the Nyquist plot produced by MATLAB Program 7-9.

MATLAB Program 7-9

```
A = [0 1;-25 -4];  
B = [0;25];  
C = [1 0];  
D = [0];  
nyquist(A,B,C,D)  
grid  
title('Nyquist Plot')
```

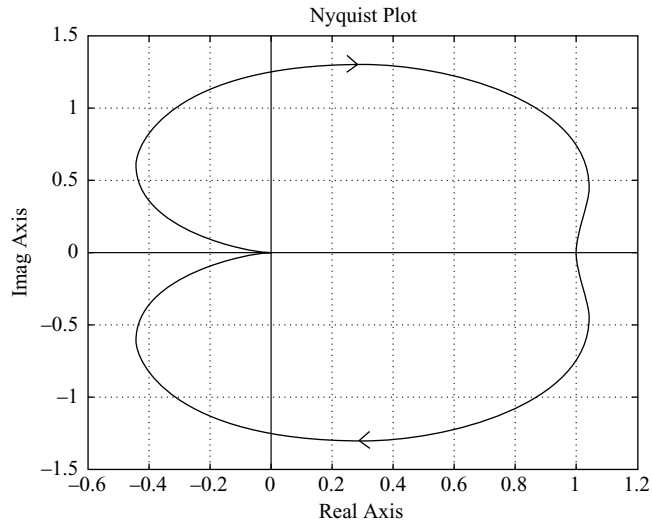


Figure 7-41
Nyquist plot of
system considered in
Example 7-12.

EXAMPLE 7-13 Consider the system defined by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 6.5 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

This system involves two inputs and two outputs. There are four sinusoidal output–input relationships: $Y_1(j\omega)/U_1(j\omega)$, $Y_2(j\omega)/U_1(j\omega)$, $Y_1(j\omega)/U_2(j\omega)$, and $Y_2(j\omega)/U_2(j\omega)$. Draw Nyquist plots for the system. (When considering input u_1 , we assume that input u_2 is zero, and vice versa.)

The four individual Nyquist plots can be obtained by the use of the command

$$\text{nyquist}(A,B,C,D)$$

MATLAB Program 7-10 produces the four Nyquist plots. They are shown in Figure 7-42.

MATLAB Program 7-10

```
A = [-1 -1;6.5 0];
B = [1 1;1 0];
C = [1 0;0 1];
D = [0 0;0 0];
nyquist(A,B,C,D)
```


Nyquist Diagrams

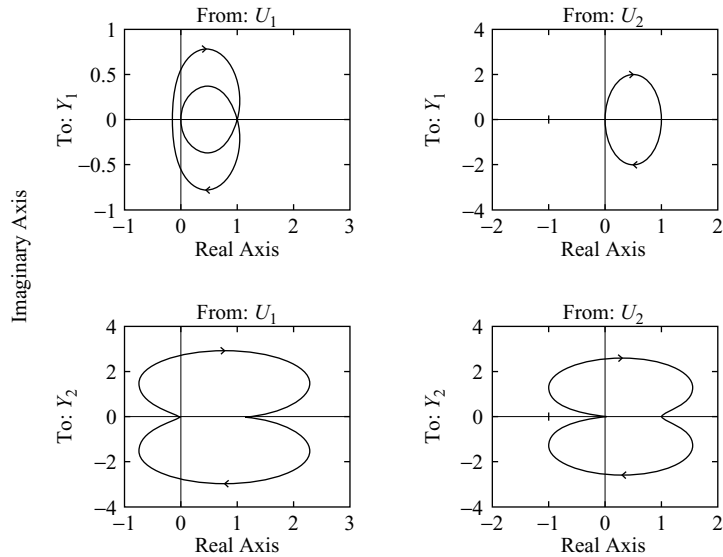


Figure 7-42
Nyquist plot of
system considered in
Example 7-13.

7-4 LOG-MAGNITUDE-VERSUS-PHASE PLOTS

Another approach to graphically portraying the frequency-response characteristics is to use the log-magnitude-versus-phase plot, which is a plot of the logarithmic magnitude in decibels versus the phase angle or phase margin for a frequency range of interest. [The phase margin is the difference between the actual phase angle ϕ and -180° ; that is, $\phi - (-180^\circ) = 180^\circ + \phi$.] The curve is graduated in terms of the frequency ω . Such log-magnitude-versus-phase plots are commonly called Nichols plots.

In the Bode diagram, the frequency-response characteristics of $G(j\omega)$ are shown on semilog paper by two separate curves, the log-magnitude curve and the phase-angle curve, while in the log-magnitude-versus-phase plot, the two curves in the Bode diagram are combined into one. In the manual approach the log-magnitude-versus-phase plot can easily be constructed by reading values of the log magnitude and phase angle from the Bode diagram. Notice that in the log-magnitude-versus-phase plot a change in the gain constant of $G(j\omega)$ merely shifts the curve up (for increasing gain) or down (for decreasing gain), but the shape of the curve remains the same.

Advantages of the log-magnitude-versus-phase plot are that the relative stability of the closed-loop system can be determined quickly and that compensation can be worked out easily.

The log-magnitude-versus-phase plot for the sinusoidal transfer function $G(j\omega)$ and that for $1/G(j\omega)$ are skew symmetrical about the origin, since

$$\left| \frac{1}{G(j\omega)} \right| \text{ in dB} = -|G(j\omega)| \text{ in dB}$$

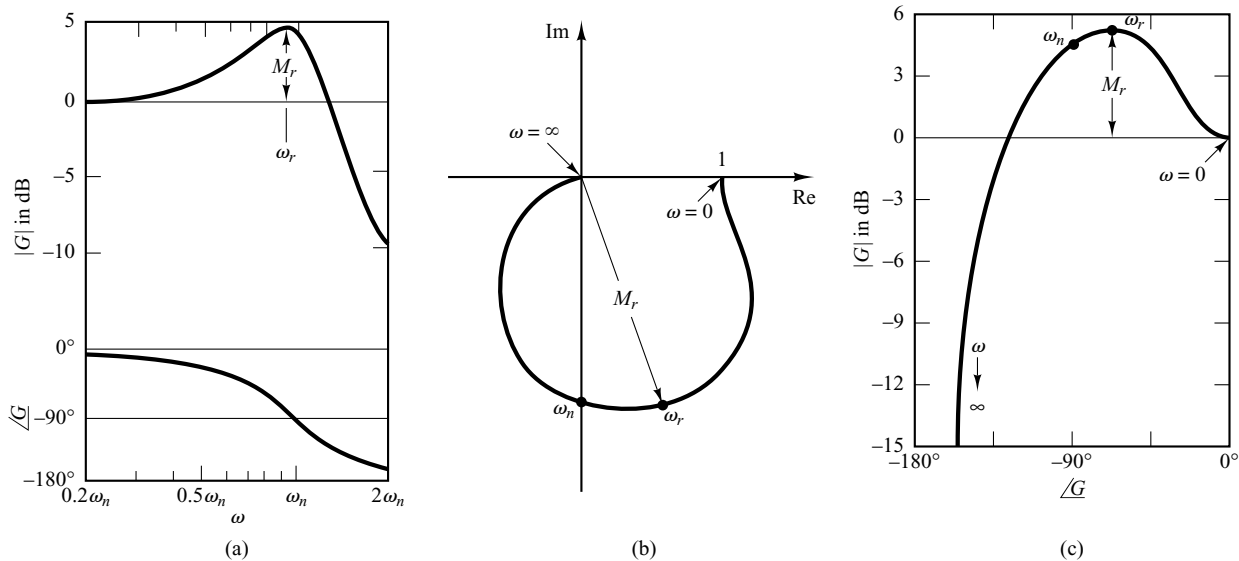


Figure 7-43

Three representations of the frequency response of $\frac{1}{1 + 2\zeta\left(j\frac{\omega}{\omega_n}\right) + \left(j\frac{\omega}{\omega_n}\right)^2}$, for $\zeta > 0$.

(a) Bode diagram; (b) polar plot; (c) log-magnitude-versus-phase plot.

and

$$\angle \frac{1}{G(j\omega)} = -\angle G(j\omega)$$

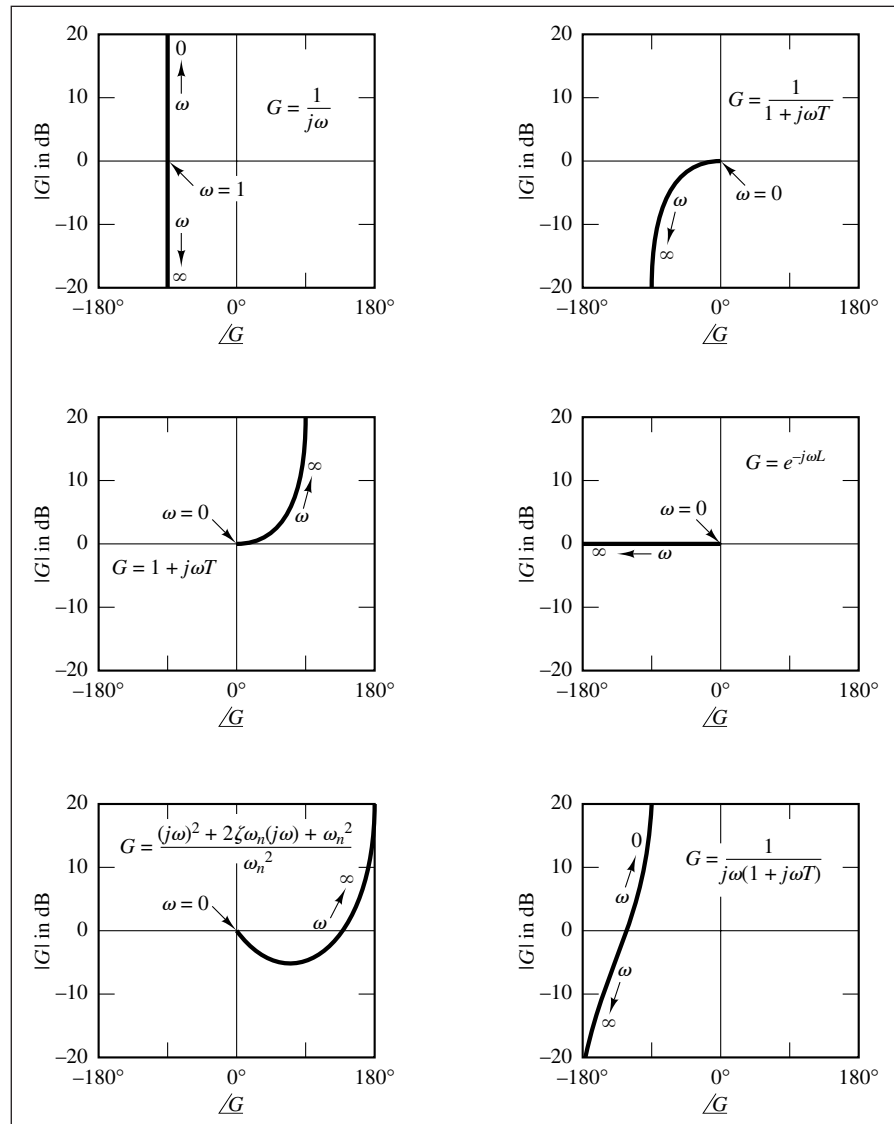
Figure 7-43 compares frequency-response curves of

$$G(j\omega) = \frac{1}{1 + 2\zeta\left(j\frac{\omega}{\omega_n}\right) + \left(j\frac{\omega}{\omega_n}\right)^2}$$

in three different representations. In the log-magnitude-versus-phase plot, the vertical distance between the points $\omega = 0$ and $\omega = \omega_r$, where ω_r is the resonant frequency, is the peak value of $G(j\omega)$ in decibels.

Since log-magnitude and phase-angle characteristics of basic transfer functions have been discussed in detail in Sections 7-2 and 7-3, it will be sufficient here to give examples of some log-magnitude-versus-phase plots. Table 7-2 shows such examples. (However, more on Nichols charts will be discussed in Section 7-6.)

Table 7-2 Log-Magnitude-versus-Phase Plots of Simple Transfer Functions



7-5 NYQUIST STABILITY CRITERION

The Nyquist stability criterion determines the stability of a closed-loop system from its open-loop frequency response and open-loop poles.

This section presents mathematical background for understanding the Nyquist stability criterion. Consider the closed-loop system shown in Figure 7-44. The closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

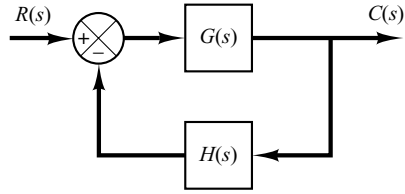


Figure 7-44
Closed-loop system.

For stability, all roots of the characteristic equation

$$1 + G(s)H(s) = 0$$

must lie in the left-half s plane. [It is noted that, although poles and zeros of the open-loop transfer function $G(s)H(s)$ may be in the right-half s plane, the system is stable if all the poles of the closed-loop transfer function (that is, the roots of the characteristic equation) are in the left-half s plane.] The Nyquist stability criterion relates the open-loop frequency response $G(j\omega)H(j\omega)$ to the number of zeros and poles of $1 + G(s)H(s)$ that lie in the right-half s plane. This criterion, derived by H. Nyquist, is useful in control engineering because the absolute stability of the closed-loop system can be determined graphically from open-loop frequency-response curves, and there is no need for actually determining the closed-loop poles. Analytically obtained open-loop frequency-response curves, as well as those experimentally obtained, can be used for the stability analysis. This is convenient because, in designing a control system, it often happens that mathematical expressions for some of the components are not known; only their frequency-response data are available.

The Nyquist stability criterion is based on a theorem from the theory of complex variables. To understand the criterion, we shall first discuss mappings of contours in the complex plane.

We shall assume that the open-loop transfer function $G(s)H(s)$ is representable as a ratio of polynomials in s . For a physically realizable system, the degree of the denominator polynomial of the closed-loop transfer function must be greater than or equal to that of the numerator polynomial. This means that the limit of $G(s)H(s)$ as s approaches infinity is zero or a constant for any physically realizable system.

Preliminary Study. The characteristic equation of the system shown in Figure 7-44 is

$$F(s) = 1 + G(s)H(s) = 0$$

We shall show that, for a given continuous closed path in the s plane that does not go through any singular points, there corresponds a closed curve in the $F(s)$ plane. The number and direction of encirclements of the origin of the $F(s)$ plane by the closed curve play a particularly important role in what follows, for later we shall correlate the number and direction of encirclements with the stability of the system.

Consider, for example, the following open-loop transfer function:

$$G(s)H(s) = \frac{2}{s-1}$$

The characteristic equation is

$$\begin{aligned} F(s) &= 1 + G(s)H(s) \\ &= 1 + \frac{2}{s-1} = \frac{s+1}{s-1} = 0 \end{aligned} \quad (7-15)$$

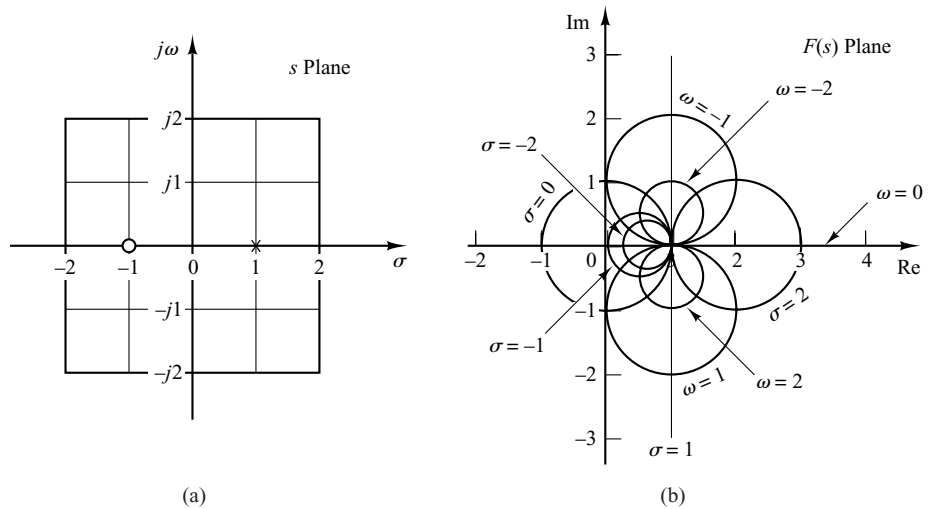


Figure 7-45
 Conformal mapping of the s -plane grids into the $F(s)$ plane, where
 $F(s) = (s + 1)/(s - 1)$.

The function $F(s)$ is analytic[#] everywhere in the s plane except at its singular points. For each point of analyticity in the s plane, there corresponds a point in the $F(s)$ plane. For example, if $s = 2 + j1$, then $F(s)$ becomes

$$F(2 + j1) = \frac{2 + j1 + 1}{2 + j1 - 1} = 2 - j1$$

Thus, point $s = 2 + j1$ in the s plane maps into point $2 - j1$ in the $F(s)$ plane.

Thus, as stated previously, for a given continuous closed path in the s plane, which does not go through any singular points, there corresponds a closed curve in the $F(s)$ plane.

For the characteristic equation $F(s)$ given by Equation (7-15), the conformal mapping of the lines $\omega = 0, \pm 1, \pm 2$ and the lines $\sigma = 0, \pm 1, \pm 2$ [see Figure 7-45(a)] yield circles in the $F(s)$ plane, as shown in Figure 7-45(b). Suppose that representative point s traces out a contour in the s plane in the clockwise direction. If the contour in the s plane encloses the pole of $F(s)$, there is one encirclement of the origin of the $F(s)$ plane by the locus of $F(s)$ in the counterclockwise direction. [See Figure 7-46(a).] If the contour in the s plane encloses the zero of $F(s)$, there is one encirclement of the origin of the $F(s)$ plane by the locus of $F(s)$ in the clockwise direction. [See Figure 7-46(b).] If the contour in the s plane encloses both the zero and the pole or if the contour encloses neither the zero nor the pole, then there is no encirclement of the origin of the $F(s)$ plane by the locus of $F(s)$. [See Figures 7-46(c) and (d).]

From the foregoing analysis, we can say that the direction of encirclement of the origin of the $F(s)$ plane by the locus of $F(s)$ depends on whether the contour in the s plane encloses a pole or a zero. Note that the location of a pole or zero in the s plane, whether in the right-half or left-half s plane, does not make any difference, but the enclosure of a pole or zero does. If the contour in the s plane encloses equal numbers of poles and zeros, then the corresponding closed curve in the $F(s)$ plane does not encircle the origin of the $F(s)$ plane. The foregoing discussion is a graphical explanation of the mapping theorem, which is the basis for the Nyquist stability criterion.

[#]A complex function $F(s)$ is said to be analytic in a region if $F(s)$ and all its derivatives exist in that region.

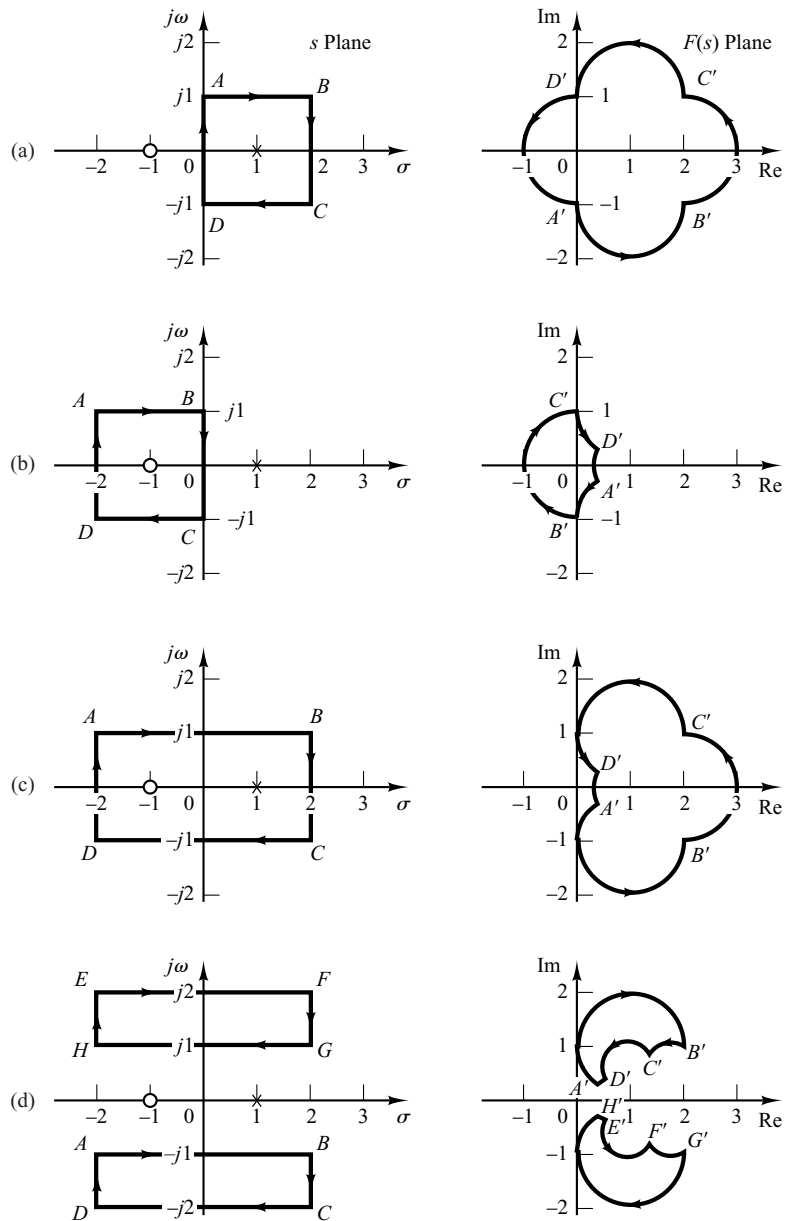


Figure 7-46

Closed contours in the s plane and their corresponding closed curves in the $F(s)$ plane, where $F(s) = (s + 1)/(s - 1)$.

Mapping Theorem. Let $F(s)$ be a ratio of two polynomials in s . Let P be the number of poles and Z be the number of zeros of $F(s)$ that lie inside some closed contour in the s plane, with multiplicity of poles and zeros accounted for. Let the contour be such that it does not pass through any poles or zeros of $F(s)$. This closed contour in the s plane is then mapped into the $F(s)$ plane as a closed curve. The total number N of clockwise encirclements of the origin of the $F(s)$ plane, as a representative point s traces out the entire contour in the clockwise direction, is equal to $Z - P$. (Note that by this mapping theorem, the numbers of zeros and of poles cannot be found—only their difference.)

We shall not present a formal proof of this theorem here, but leave the proof to Problem A-7-6. Note that a positive number N indicates an excess of zeros over poles of the function $F(s)$ and a negative N indicates an excess of poles over zeros. In control system applications, the number P can be readily determined for $F(s) = 1 + G(s)H(s)$ from the function $G(s)H(s)$. Therefore, if N is determined from the plot of $F(s)$, the number of zeros in the closed contour in the s plane can be determined readily. Note that the exact shapes of the s -plane contour and $F(s)$ locus are immaterial so far as encirclements of the origin are concerned, since encirclements depend only on the enclosure of poles and/or zeros of $F(s)$ by the s -plane contour.

Application of the Mapping Theorem to the Stability Analysis of Closed-Loop Systems. For analyzing the stability of linear control systems, we let the closed contour in the s plane enclose the entire right-half s plane. The contour consists of the entire $j\omega$ axis from $\omega = -\infty$ to $+\infty$ and a semicircular path of infinite radius in the right-half s plane. Such a contour is called the Nyquist path. (The direction of the path is clockwise.) The Nyquist path encloses the entire right-half s plane and encloses all the zeros and poles of $1 + G(s)H(s)$ that have positive real parts. [If there are no zeros of $1 + G(s)H(s)$ in the right-half s plane, then there are no closed-loop poles there, and the system is stable.] It is necessary that the closed contour, or the Nyquist path, not pass through any zeros and poles of $1 + G(s)H(s)$. If $G(s)H(s)$ has a pole or poles at the origin of the s plane, mapping of the point $s = 0$ becomes indeterminate. In such cases, the origin is avoided by taking a detour around it. (A detailed discussion of this special case is given later.)

If the mapping theorem is applied to the special case in which $F(s)$ is equal to $1 + G(s)H(s)$, then we can make the following statement: If the closed contour in the s plane encloses the entire right-half s plane, as shown in Figure 7-47, then the number of right-half plane zeros of the function $F(s) = 1 + G(s)H(s)$ is equal to the number of poles of the function $F(s) = 1 + G(s)H(s)$ in the right-half s plane plus the number of clockwise encirclements of the origin of the $1 + G(s)H(s)$ plane by the corresponding closed curve in this latter plane.

Because of the assumed condition that

$$\lim_{s \rightarrow \infty} [1 + G(s)H(s)] = \text{constant}$$

the function of $1 + G(s)H(s)$ remains constant as s traverses the semicircle of infinite radius. Because of this, whether the locus of $1 + G(s)H(s)$ encircles the origin of the $1 + G(s)H(s)$ plane can be determined by considering only a part of the closed contour in the s plane—that is, the $j\omega$ axis. Encirclements of the origin, if there are any, occur only

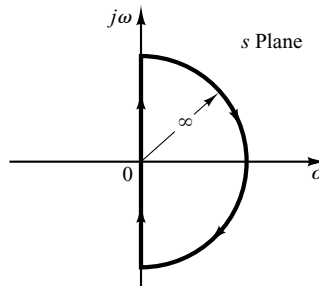
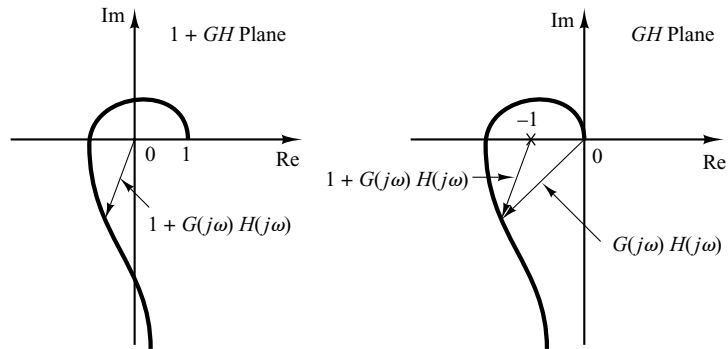


Figure 7-47
Closed contour in
the s plane.

Figure 7–48
Plots of
 $1 + G(j\omega)H(j\omega)$ in
the $1 + GH$ plane
and GH plane.



while a representative point moves from $-j\infty$ to $+j\infty$ along the $j\omega$ axis, provided that no zeros or poles lie on the $j\omega$ axis.

Note that the portion of the $1 + G(s)H(s)$ contour from $\omega = -\infty$ to $\omega = \infty$ is simply $1 + G(j\omega)H(j\omega)$. Since $1 + G(j\omega)H(j\omega)$ is the vector sum of the unit vector and the vector $G(j\omega)H(j\omega)$, $1 + G(j\omega)H(j\omega)$ is identical to the vector drawn from the $-1 + j0$ point to the terminal point of the vector $G(j\omega)H(j\omega)$, as shown in Figure 7–48. Encirclement of the origin by the graph of $1 + G(j\omega)H(j\omega)$ is equivalent to encirclement of the $-1 + j0$ point by just the $G(j\omega)H(j\omega)$ locus. Thus, the stability of a closed-loop system can be investigated by examining encirclements of the $-1 + j0$ point by the locus of $G(j\omega)H(j\omega)$. The number of clockwise encirclements of the $-1 + j0$ point can be found by drawing a vector from the $-1 + j0$ point to the $G(j\omega)H(j\omega)$ locus, starting from $\omega = -\infty$, going through $\omega = 0$, and ending at $\omega = +\infty$, and by counting the number of clockwise rotations of the vector.

Plotting $G(j\omega)H(j\omega)$ for the Nyquist path is straightforward. The map of the negative $j\omega$ axis is the mirror image about the real axis of the map of the positive $j\omega$ axis. That is, the plot of $G(j\omega)H(j\omega)$ and the plot of $G(-j\omega)H(-j\omega)$ are symmetrical with each other about the real axis. The semicircle with infinite radius maps into either the origin of the GH plane or a point on the real axis of the GH plane.

In the preceding discussion, $G(s)H(s)$ has been assumed to be the ratio of two polynomials in s . Thus, the transport lag e^{-Ts} has been excluded from the discussion. Note, however, that a similar discussion applies to systems with transport lag, although a proof of this is not given here. The stability of a system with transport lag can be determined from the open-loop frequency-response curves by examining the number of encirclements of the $-1 + j0$ point, just as in the case of a system whose open-loop transfer function is a ratio of two polynomials in s .

Nyquist Stability Criterion. The foregoing analysis, utilizing the encirclement of the $-1 + j0$ point by the $G(j\omega)H(j\omega)$ locus, is summarized in the following Nyquist stability criterion:

Nyquist stability criterion [for a special case when $G(s)H(s)$ has neither poles nor zeros on the $j\omega$ axis]: In the system shown in Figure 7–44, if the open-loop transfer function $G(s)H(s)$ has k poles in the right-half s plane and $\lim_{s \rightarrow \infty} G(s)H(s) = \text{constant}$, then for stability, the $G(j\omega)H(j\omega)$ locus, as ω varies from $-\infty$ to ∞ , must encircle the $-1 + j0$ point k times in the counterclockwise direction.

Remarks on the Nyquist Stability Criterion

1. This criterion can be expressed as

$$Z = N + P$$

where Z = number of zeros of $1 + G(s)H(s)$ in the right-half s plane
 N = number of clockwise encirclements of the $-1 + j0$ point
 P = number of poles of $G(s)H(s)$ in the right-half s plane

If P is not zero, for a stable control system, we must have $Z = 0$, or $N = -P$, which means that we must have P counterclockwise encirclements of the $-1 + j0$ point.

If $G(s)H(s)$ does not have any poles in the right-half s plane, then $Z = N$. Thus, for stability there must be no encirclement of the $-1 + j0$ point by the $G(j\omega)H(j\omega)$ locus. In this case it is not necessary to consider the locus for the entire $j\omega$ axis, only for the positive-frequency portion. The stability of such a system can be determined by seeing if the $-1 + j0$ point is enclosed by the Nyquist plot of $G(j\omega)H(j\omega)$. The region enclosed by the Nyquist plot is shown in Figure 7-49. For stability, the $-1 + j0$ point must lie outside the shaded region.

2. We must be careful when testing the stability of multiple-loop systems since they may include poles in the right-half s plane. (Note that although an inner loop may be unstable, the entire closed-loop system can be made stable by proper design.) Simple inspection of encirclements of the $-1 + j0$ point by the $G(j\omega)H(j\omega)$ locus is not sufficient to detect instability in multiple-loop systems. In such cases, however, whether any pole of $1 + G(s)H(s)$ is in the right-half s plane can be determined easily by applying the Routh stability criterion to the denominator of $G(s)H(s)$.

If transcendental functions, such as transport lag e^{-Ts} , are included in $G(s)H(s)$, they must be approximated by a series expansion before the Routh stability criterion can be applied.

3. If the locus of $G(j\omega)H(j\omega)$ passes through the $-1 + j0$ point, then zeros of the characteristic equation, or closed-loop poles, are located on the $j\omega$ axis. This is not desirable for practical control systems. For a well-designed closed-loop system, none of the roots of the characteristic equation should lie on the $j\omega$ axis.

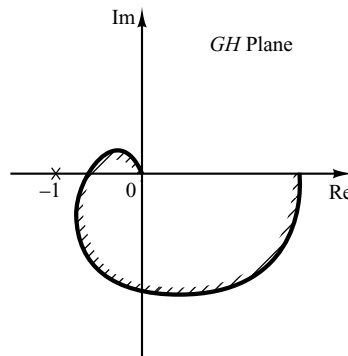


Figure 7-49
Region enclosed by a
Nyquist plot.

Special Case when $G(s)H(s)$ Involves Poles and/or Zeros on the $j\omega$ Axis. In the previous discussion, we assumed that the open-loop transfer function $G(s)H(s)$ has neither poles nor zeros at the origin. We now consider the case where $G(s)H(s)$ involves poles and/or zeros on the $j\omega$ axis.

Since the Nyquist path must not pass through poles or zeros of $G(s)H(s)$, if the function $G(s)H(s)$ has poles or zeros at the origin (or on the $j\omega$ axis at points other than the origin), the contour in the s plane must be modified. The usual way of modifying the contour near the origin is to use a semicircle with the infinitesimal radius ϵ , as shown in Figure 7–50. [Note that this semicircle may lie in the right-half s plane or in the left-half s plane. Here we take the semicircle in the right-half s plane.] A representative point s moves along the negative $j\omega$ axis from $-j\infty$ to $j0-$. From $s = j0-$ to $s = j0+$, the point moves along the semicircle of radius ϵ (where $\epsilon \ll 1$) and then moves along the positive $j\omega$ axis from $j0+$ to $j\infty$. From $s = j\infty$, the contour follows a semicircle with infinite radius, and the representative point moves back to the starting point, $s = -j\infty$. The area that the modified closed contour avoids is very small and approaches zero as the radius ϵ approaches zero. Therefore, all the poles and zeros, if any, in the right-half s plane are enclosed by this contour.

Consider, for example, a closed-loop system whose open-loop transfer function is given by

$$G(s)H(s) = \frac{K}{s(Ts + 1)}$$

The points corresponding to $s = j0+$ and $s = j0-$ on the locus of $G(s)H(s)$ in the $G(s)H(s)$ plane are $-j\infty$ and $j\infty$, respectively. On the semicircular path with radius ϵ (where $\epsilon \ll 1$), the complex variable s can be written

$$s = \epsilon e^{j\theta}$$

where θ varies from -90° to $+90^\circ$. Then $G(s)H(s)$ becomes

$$G(\epsilon e^{j\theta})H(\epsilon e^{j\theta}) = \frac{K}{\epsilon e^{j\theta}} = \frac{K}{\epsilon} e^{-j\theta}$$

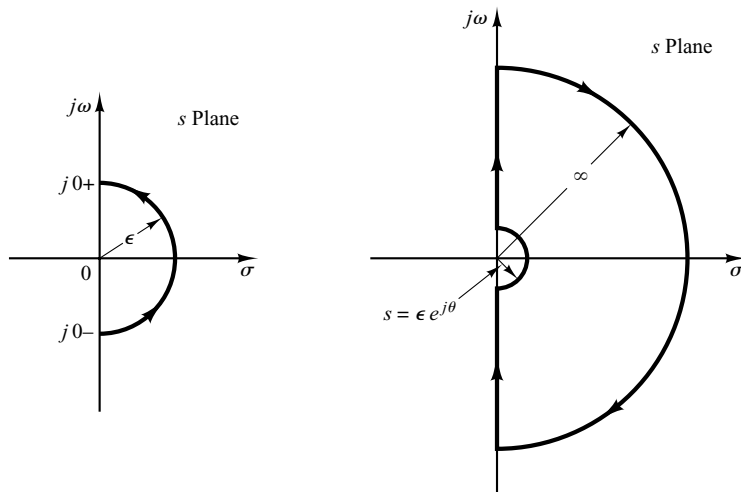


Figure 7–50
Contour near the origin of the s plane and closed contour in the s plane avoiding poles and zeros at the origin.

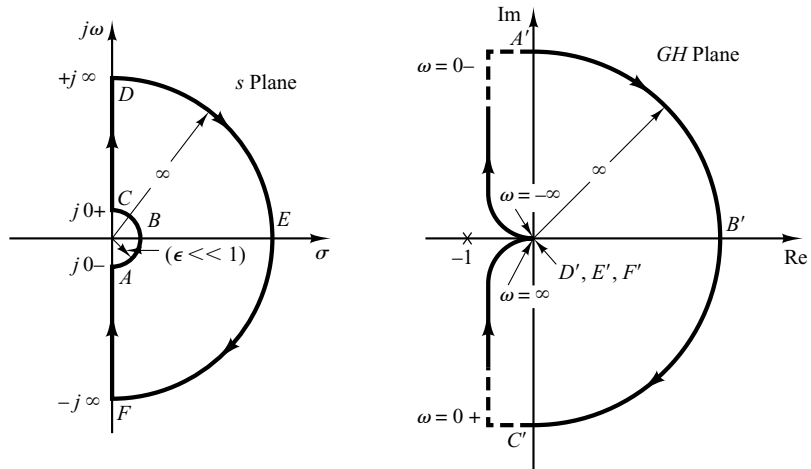


Figure 7-51
s-Plane contour and the $G(s)H(s)$ locus in the GH plane, where $G(s)H(s) = K/[s(Ts + 1)]$.

The value K/ϵ approaches infinity as ϵ approaches zero, and $-\theta$ varies from 90° to -90° as a representative point s moves along the semicircle in the s plane. Thus, the points $G(j0^-)H(j0^-) = j\infty$ and $G(j0^+)H(j0^+) = -j\infty$ are joined by a semicircle of infinite radius in the right-half GH plane. The infinitesimal semicircular detour around the origin in the s plane maps into the GH plane as a semicircle of infinite radius. Figure 7-51 shows the s -plane contour and the $G(s)H(s)$ locus in the GH plane. Points A , B , and C on the s -plane contour map into the respective points A' , B' , and C' on the $G(s)H(s)$ locus. As seen from Figure 7-51, points D , E , and F on the semicircle of infinite radius in the s plane map into the origin of the GH plane. Since there is no pole in the right-half s plane and the $G(s)H(s)$ locus does not encircle the $-1 + j0$ point, there are no zeros of the function $1 + G(s)H(s)$ in the right-half s plane. Therefore, the system is stable.

For an open-loop transfer function $G(s)H(s)$ involving a $1/s^n$ factor (where $n = 2, 3, \dots$), the plot of $G(s)H(s)$ has n clockwise semicircles of infinite radius about the origin as a representative point s moves along the semicircle of radius ϵ (where $\epsilon \ll 1$). For example, consider the following open-loop transfer function:

$$G(s)H(s) = \frac{K}{s^2(Ts + 1)}$$

Then

$$\lim_{s \rightarrow \epsilon e^{j\theta}} G(s)H(s) = \frac{K}{\epsilon^2 e^{2j\theta}} = \frac{K}{\epsilon^2} e^{-2j\theta}$$

As θ varies from -90° to 90° in the s plane, the angle of $G(s)H(s)$ varies from 180° to -180° , as shown in Figure 7-52. Since there is no pole in the right-half s plane and the locus encircles the $-1 + j0$ point twice clockwise for any positive value of K , there are two zeros of $1 + G(s)H(s)$ in the right-half s plane. Therefore, this system is always unstable.

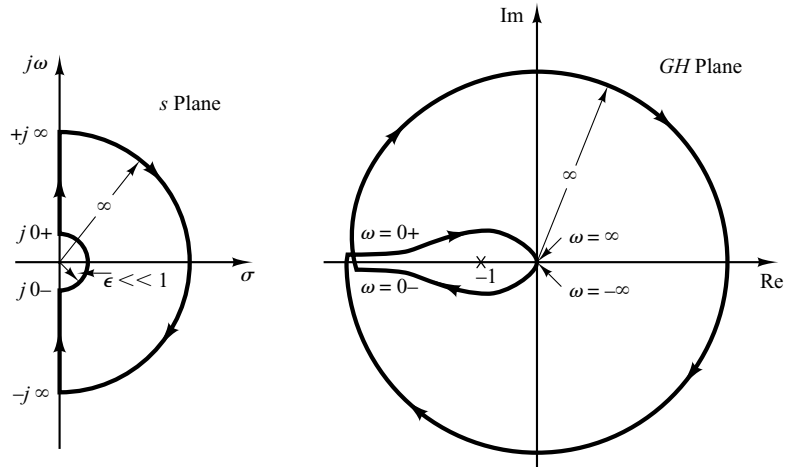


Figure 7-52
 s -Plane contour and the $G(s)H(s)$ locus in the GH plane, where $G(s)H(s) = K/[s^2(Ts + 1)]$.

Note that a similar analysis can be made if $G(s)H(s)$ involves poles and/or zeros on the $j\omega$ axis. The Nyquist stability criterion can now be generalized as follows:

Nyquist stability criterion [for a general case when $G(s)H(s)$ has poles and/or zeros on the $j\omega$ axis]: In the system shown in Figure 7-44, if the open-loop transfer function $G(s)H(s)$ has k poles in the right-half s plane, then for stability the $G(s)H(s)$ locus, as a representative point s traces on the modified Nyquist path in the clockwise direction, must encircle the $-1 + j0$ point k times in the counterclockwise direction.

7-6 STABILITY ANALYSIS

In this section, we shall present several illustrative examples of the stability analysis of control systems using the Nyquist stability criterion.

If the Nyquist path in the s plane encircles Z zeros and P poles of $1 + G(s)H(s)$ and does not pass through any poles or zeros of $1 + G(s)H(s)$ as a representative point s moves in the clockwise direction along the Nyquist path, then the corresponding contour in the $G(s)H(s)$ plane encircles the $-1 + j0$ point $N = Z - P$ times in the clockwise direction. (Negative values of N imply counterclockwise encirclements.)

In examining the stability of linear control systems using the Nyquist stability criterion, we see that three possibilities can occur:

1. There is no encirclement of the $-1 + j0$ point. This implies that the system is stable if there are no poles of $G(s)H(s)$ in the right-half s plane; otherwise, the system is unstable.
2. There are one or more counterclockwise encirclements of the $-1 + j0$ point. In this case the system is stable if the number of counterclockwise encirclements is the same as the number of poles of $G(s)H(s)$ in the right-half s plane; otherwise, the system is unstable.
3. There are one or more clockwise encirclements of the $-1 + j0$ point. In this case the system is unstable.

In the following examples, we assume that the values of the gain K and the time constants (such as T , T_1 , and T_2) are all positive.

EXAMPLE 7-14 Consider a closed-loop system whose open-loop transfer function is given by

$$G(s)H(s) = \frac{K}{(T_1s + 1)(T_2s + 1)}$$

Examine the stability of the system.

A plot of $G(j\omega)H(j\omega)$ is shown in Figure 7-53. Since $G(s)H(s)$ does not have any poles in the right-half s plane and the $-1 + j0$ point is not encircled by the $G(j\omega)H(j\omega)$ locus, this system is stable for any positive values of $K, T_1,$ and T_2 .

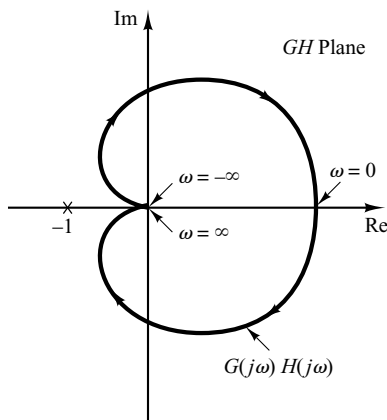


Figure 7-53
Polar plot of $G(j\omega)H(j\omega)$ considered in Example 7-14.

EXAMPLE 7-15 Consider the system with the following open-loop transfer function:

$$G(s)H(s) = \frac{K}{s(T_1s + 1)(T_2s + 1)}$$

Determine the stability of the system for two cases: (1) the gain K is small and (2) K is large.

The Nyquist plots of the open-loop transfer function with a small value of K and a large value of K are shown in Figure 7-54. The number of poles of $G(s)H(s)$ in the right-half s plane is zero.

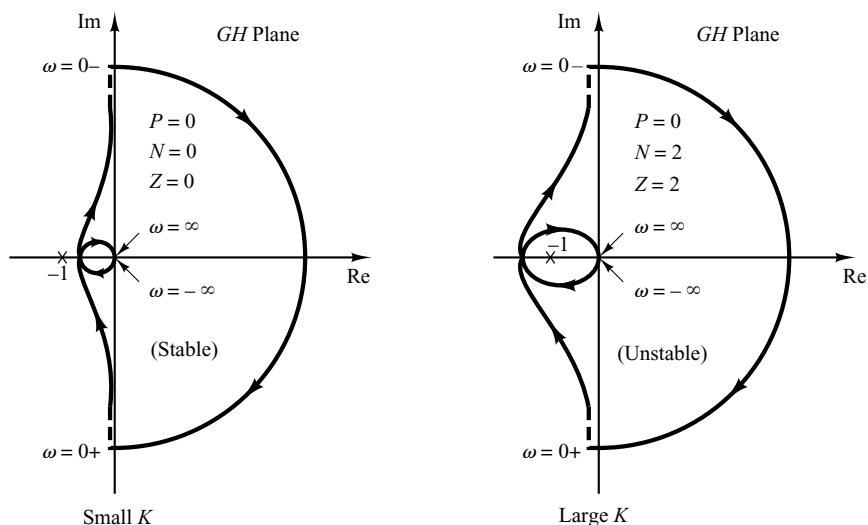


Figure 7-54
Polar plots of the system considered in Example 7-15.

Therefore, for this system to be stable, it is necessary that $N = Z = 0$ or that the $G(s)H(s)$ locus not encircle the $-1 + j0$ point.

For small values of K , there is no encirclement of the $-1 + j0$ point. Hence, the system is stable for small values of K . For large values of K , the locus of $G(s)H(s)$ encircles the $-1 + j0$ point twice in the clockwise direction, indicating two closed-loop poles in the right-half s plane, and the system is unstable. (For good accuracy, K should be large. From the stability viewpoint, however, a large value of K causes poor stability or even instability. To compromise between accuracy and stability, it is necessary to insert a compensation network into the system. Compensating techniques in the frequency domain are discussed in Sections 7–11 through 7–13.)

EXAMPLE 7–16 The stability of a closed-loop system with the following open-loop transfer function

$$G(s)H(s) = \frac{K(T_2s + 1)}{s^2(T_1s + 1)}$$

depends on the relative magnitudes of T_1 and T_2 . Draw Nyquist plots and determine the stability of the system.

Plots of the locus $G(s)H(s)$ for three cases, $T_1 < T_2$, $T_1 = T_2$, and $T_1 > T_2$, are shown in Figure 7–55. For $T_1 < T_2$, the locus of $G(s)H(s)$ does not encircle the $-1 + j0$ point, and the closed-loop system is stable. For $T_1 = T_2$, the $G(s)H(s)$ locus passes through the $-1 + j0$ point, which indicates that the closed-loop poles are located on the $j\omega$ axis. For $T_1 > T_2$, the locus of $G(s)H(s)$ encircles the $-1 + j0$ point twice in the clockwise direction. Thus, the closed-loop system has two closed-loop poles in the right-half s plane, and the system is unstable.

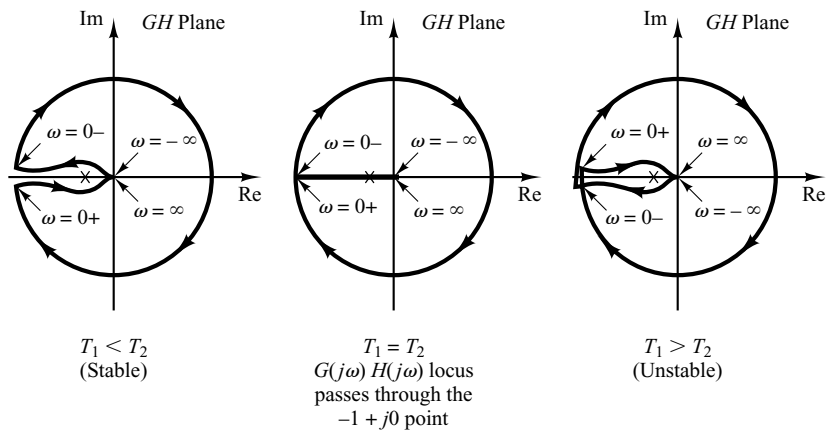


Figure 7–55
Polar plots of the system considered in Example 7–16.

EXAMPLE 7–17 Consider the closed-loop system having the following open-loop transfer function:

$$G(s)H(s) = \frac{K}{s(Ts - 1)}$$

Determine the stability of the system.

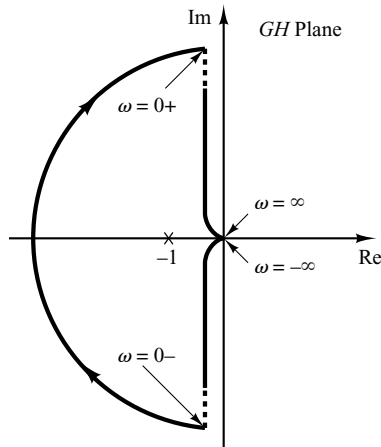


Figure 7-56
Polar plot of the system considered in Example 7-17.

The function $G(s)H(s)$ has one pole ($s = 1/T$) in the right-half s plane. Therefore, $P = 1$. The Nyquist plot shown in Figure 7-56 indicates that the $G(s)H(s)$ plot encircles the $-1 + j0$ point once clockwise. Thus, $N = 1$. Since $Z = N + P$, we find that $Z = 2$. This means that the closed-loop system has two closed-loop poles in the right-half s plane and is unstable.

EXAMPLE 7-18 Investigate the stability of a closed-loop system with the following open-loop transfer function:

$$G(s)H(s) = \frac{K(s + 3)}{s(s - 1)} \quad (K > 1)$$

The open-loop transfer function has one pole ($s = 1$) in the right-half s plane, or $P = 1$. The open-loop system is unstable. The Nyquist plot shown in Figure 7-57 indicates that the $-1 + j0$ point is encircled by the $G(s)H(s)$ locus once in the counterclockwise direction. Therefore, $N = -1$. Thus, Z is found from $Z = N + P$ to be zero, which indicates that there is no zero of $1 + G(s)H(s)$ in the right-half s plane, and the closed-loop system is stable. This is one of the examples for which an unstable open-loop system becomes stable when the loop is closed.

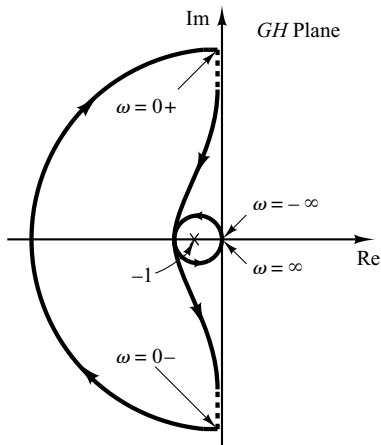


Figure 7-57
Polar plot of the system considered in Example 7-18.

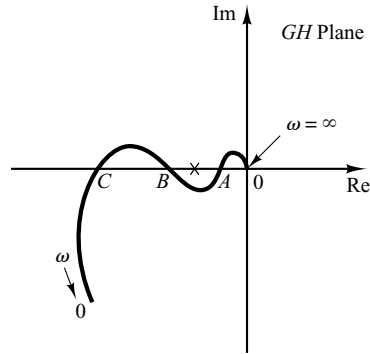


Figure 7–58
Polar plot of a
conditionally stable
system.

Conditionally Stable Systems. Figure 7–58 shows an example of a $G(j\omega)H(j\omega)$ locus for which the closed-loop system can be made unstable by varying the open-loop gain. If the open-loop gain is increased sufficiently, the $G(j\omega)H(j\omega)$ locus encloses the $-1 + j0$ point twice, and the system becomes unstable. If the open-loop gain is decreased sufficiently, again the $G(j\omega)H(j\omega)$ locus encloses the $-1 + j0$ point twice. For stable operation of the system considered here, the critical point $-1 + j0$ must not be located in the regions between OA and BC shown in Figure 7–58. Such a system that is stable only for limited ranges of values of the open-loop gain for which the $-1 + j0$ point is completely outside the $G(j\omega)H(j\omega)$ locus is a conditionally stable system.

A conditionally stable system is stable for the value of the open-loop gain lying between critical values, but it is unstable if the open-loop gain is either increased or decreased sufficiently. Such a system becomes unstable when large input signals are applied, since a large signal may cause saturation, which in turn reduces the open-loop gain of the system. It is advisable to avoid such a situation.

Multiple-Loop System. Consider the system shown in Figure 7–59. This is a multiple-loop system. The inner loop has the transfer function

$$G(s) = \frac{G_2(s)}{1 + G_2(s)H_2(s)}$$

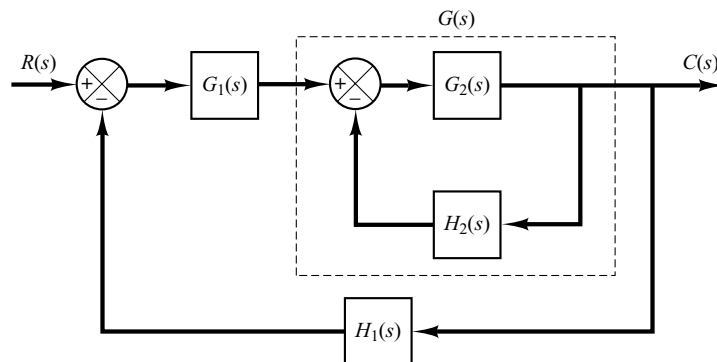


Figure 7–59
Multiple-loop
system.

If $G(s)$ is unstable, the effects of instability are to produce a pole or poles in the right-half s plane. Then the characteristic equation of the inner loop, $1 + G_2(s)H_2(s) = 0$, has a zero or zeros in the right-half s plane. If $G_2(s)$ and $H_2(s)$ have P_1 poles here, then the number Z_1 of right-half plane zeros of $1 + G_2(s)H_2(s)$ can be found from $Z_1 = N_1 + P_1$, where N_1 is the number of clockwise encirclements of the $-1 + j0$ point by the $G_2(s)H_2(s)$ locus. Since the open-loop transfer function of the entire system is given by $G_1(s)G(s)H_1(s)$, the stability of this closed-loop system can be found from the Nyquist plot of $G_1(s)G(s)H_1(s)$ and knowledge of the right-half plane poles of $G_1(s)G(s)H_1(s)$.

Notice that if a feedback loop is eliminated by means of block diagram reductions, there is a possibility that unstable poles are introduced; if the feedforward branch is eliminated by means of block diagram reductions, there is a possibility that right-half plane zeros are introduced. Therefore, we must note all right-half plane poles and zeros as they appear from subsidiary loop reductions. This knowledge is necessary in determining the stability of multiple-loop systems.

EXAMPLE 7-19

Consider the control system shown in Figure 7-60. The system involves two loops. Determine the range of gain K for stability of the system by the use of the Nyquist stability criterion. (The gain K is positive.)

To examine the stability of the control system, we need to sketch the Nyquist locus of $G(s)$, where

$$G(s) = G_1(s)G_2(s)$$

However, the poles of $G(s)$ are not known at this point. Therefore, we need to examine the minor loop if there are right-half s -plane poles. This can be done easily by use of the Routh stability criterion. Since

$$G_2(s) = \frac{1}{s^3 + s^2 + 1}$$

the Routh array becomes as follows:

$$\begin{array}{ccc} s^3 & 1 & 0 \\ s^2 & 1 & 1 \\ s^1 & -1 & 0 \\ s^0 & 1 & \end{array}$$

Notice that there are two sign changes in the first column. Hence, there are two poles of $G_2(s)$ in the right-half s plane.

Once we find the number of right-half s plane poles of $G_2(s)$, we proceed to sketch the Nyquist locus of $G(s)$, where

$$G(s) = G_1(s)G_2(s) = \frac{K(s + 0.5)}{s^3 + s^2 + 1}$$

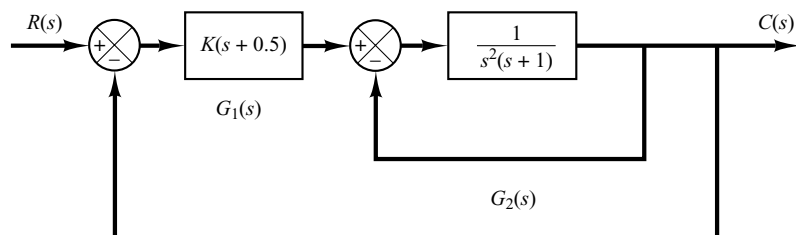


Figure 7-60
Control system.

Our problem is to determine the range of the gain K for stability. Hence, instead of plotting Nyquist loci of $G(j\omega)$ for various values of K , we plot the Nyquist locus of $G(j\omega)/K$. Figure 7-61 shows the Nyquist plot or polar plot of $G(j\omega)/K$.

Since $G(s)$ has two poles in the right-half s plane, we have $P = 2$. Noting that

$$Z = N + P$$

for stability, we require $Z = 0$ or $N = -2$. That is, the Nyquist locus of $G(j\omega)$ must encircle the $-1 + j0$ point twice counterclockwise. From Figure 7-61, we see that, if the critical point lies between 0 and -0.5 , then the $G(j\omega)/K$ locus encircles the critical point twice counterclockwise. Therefore, we require

$$-0.5K < -1$$

The range of the gain K for stability is

$$2 < K$$

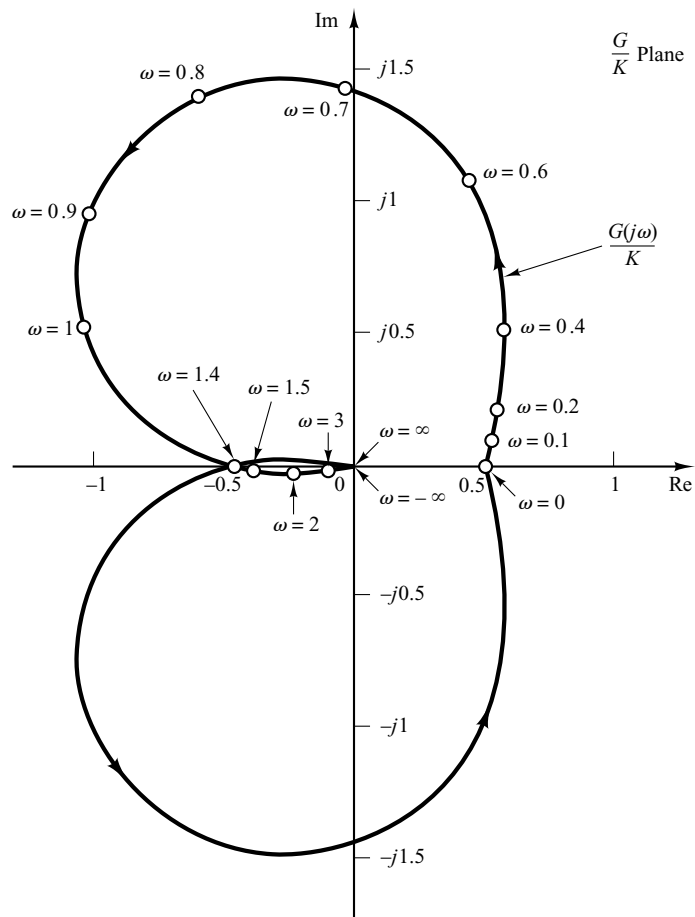


Figure 7-61
Polar plot of
 $G(j\omega)/K$.

Nyquist Stability Criterion Applied to Inverse Polar Plots. In the previous analyses, the Nyquist stability criterion was applied to polar plots of the open-loop transfer function $G(s)H(s)$.

In analyzing multiple-loop systems, the inverse transfer function may sometimes be used in order to permit graphical analysis; this avoids much of the numerical calculation. (The Nyquist stability criterion can be applied equally well to inverse polar plots. The mathematical derivation of the Nyquist stability criterion for inverse polar plots is the same as that for direct polar plots.)

The inverse polar plot of $G(j\omega)H(j\omega)$ is a graph of $1/[G(j\omega)H(j\omega)]$ as a function of ω . For example, if $G(j\omega)H(j\omega)$ is

$$G(j\omega)H(j\omega) = \frac{j\omega T}{1 + j\omega T}$$

then

$$\frac{1}{G(j\omega)H(j\omega)} = \frac{1}{j\omega T} + 1$$

The inverse polar plot for $\omega \geq 0$ is the lower half of the vertical line starting at the point $(1, 0)$ on the real axis.

The Nyquist stability criterion applied to inverse plots may be stated as follows: For a closed-loop system to be stable, the encirclement, if any, of the $-1 + j0$ point by the $1/[G(s)H(s)]$ locus (as s moves along the Nyquist path) must be counterclockwise, and the number of such encirclements must be equal to the number of poles of $1/[G(s)H(s)]$ [that is, the zeros of $G(s)H(s)$] that lie in the right-half s plane. [The number of zeros of $G(s)H(s)$ in the right-half s plane may be determined by the use of the Routh stability criterion.] If the open-loop transfer function $G(s)H(s)$ has no zeros in the right-half s plane, then for a closed-loop system to be stable, the number of encirclements of the $-1 + j0$ point by the $1/[G(s)H(s)]$ locus must be zero.

Note that although the Nyquist stability criterion can be applied to inverse polar plots, if experimental frequency-response data are incorporated, counting the number of encirclements of the $1/[G(s)H(s)]$ locus may be difficult because the phase shift corresponding to the infinite semicircular path in the s plane is difficult to measure. For example, if the open-loop transfer function $G(s)H(s)$ involves transport lag such that

$$G(s)H(s) = \frac{Ke^{-j\omega L}}{s(Ts + 1)}$$

then the number of encirclements of the $-1 + j0$ point by the $1/[G(s)H(s)]$ locus becomes infinite, and the Nyquist stability criterion cannot be applied to the inverse polar plot of such an open-loop transfer function.

In general, if experimental frequency-response data cannot be put into analytical form, both the $G(j\omega)H(j\omega)$ and $1/[G(j\omega)H(j\omega)]$ loci must be plotted. In addition, the number of right-half plane zeros of $G(s)H(s)$ must be determined. It is more difficult to determine the right-half plane zeros of $G(s)H(s)$ (in other words, to determine whether a given component is minimum phase) than it is to determine the right-half plane poles of $G(s)H(s)$ (in other words, to determine whether the component is stable).

Depending on whether the data are graphical or analytical and whether nonminimum-phase components are included, an appropriate stability test must be used for multiple-loop systems. If the data are given in analytical form or if mathematical expressions for all the components are known, the application of the Nyquist stability criterion to inverse polar plots causes no difficulty, and multiple-loop systems may be analyzed and designed in the inverse GH plane. (See Problem A-7-15.)

7-7 RELATIVE STABILITY ANALYSIS

Relative Stability. In designing a control system, we require that the system be stable. Furthermore, it is necessary that the system have adequate relative stability.

In this section, we shall show that the Nyquist plot indicates not only whether a system is stable, but also the degree of stability of a stable system. The Nyquist plot also gives information as to how stability may be improved, if this is necessary.

In the following discussion, we shall assume that the systems considered have unity feedback. Note that it is always possible to reduce a system with feedback elements to a unity-feedback system, as shown in Figure 7-62. Hence, the extension of relative stability analysis for the unity-feedback system to nonunity-feedback systems is possible.

We shall also assume that, unless otherwise stated, the systems are minimum-phase systems; that is, the open-loop transfer function has neither poles nor zeros in the right-half s plane.

Relative Stability Analysis by Conformal Mapping. One of the important problems in analyzing a control system is to find all closed-loop poles or at least those closest to the $j\omega$ axis (or the dominant pair of closed-loop poles). If the open-loop frequency-response characteristics of a system are known, it may be possible to estimate the closed-loop poles closest to the $j\omega$ axis. It is noted that the Nyquist locus $G(j\omega)$ need not be an analytically known function of ω . The entire Nyquist locus may be experimentally obtained. The technique to be presented here is essentially graphical and is based on a conformal mapping of the s plane into the $G(s)$ plane.

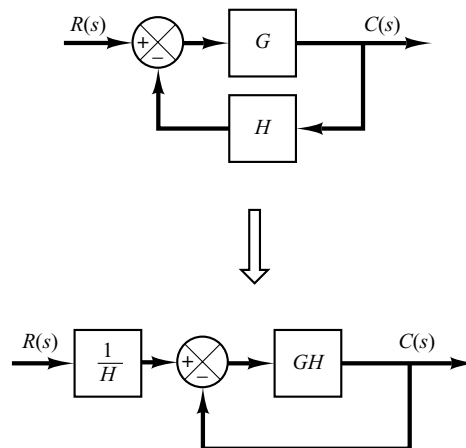


Figure 7-62
Modification of a system with feedback elements to a unity-feedback system.