We now turn to the phase plot. Table 10.7 is formed to determine the progression of slopes on the phase diagram. The first-order pole at -2 yields a phase angle that starts at 0° and ends at -90° via a -45°/decade slope starting a decade below its break frequency and ending a decade above its break frequency. The first-order zero yields a phase angle that starts at 0° and ends at $+90^\circ$ via a $+45^\circ$/decade slope starting a decade below its break frequency and ending a decade above its break frequency. The second-order poles yield a phase angle that starts at 0° and ends at -180° via a -90°/decade slope starting a decade below their natural frequency ($\omega_n = 5$) and ending a decade above their natural frequency. The slopes, shown in Figure 10.19(a), are summed over each frequency range, and the final Bode phase plot is shown in Figure 10.19(b).

MATLAB ML

```
Students who are using MATLAB should now run ch10p1 in Appendix B.
You will learn how to use MATLAB to make Bode plots and list the
points on the plots. This exercise solves Example 10.3 using MATLAB.
```

Skill-Assessment Exercise 10.2

PROBLEM: Draw the Bode log-magnitude and phase plots for the system shown in Figure 10.10, where

$$G(s) = \frac{(s+20)}{(s+1)(s+7)(s+50)}$$

WileyPLUS WPCS Control Solutions

ANSWER: The complete solution is at www.wiley.com/college/nise.

TryIt 10.1

Use MATLAB, the Control System Toolbox, and the following statements to obtain the Bode plots for the system of Skill-Assessment Exercise 10.2

```
G=zpk([-20],[-1,-7,...
 -50],1)
bode(G);grid on
```

After the Bode plots appear, click on the curve and drag to read the coordinates.

In this section, we learned how to construct Bode log-magnitude and Bode phase plots. The Bode plots are separate magnitude and phase frequency response curves for a system, $G(s)$. In the next section, we develop the Nyquist criterion for stability, which makes use of the frequency response of a system. The Bode plots can then be used to determine the stability of a system.

10.3 Introduction to the Nyquist Criterion

The Nyquist criterion relates the stability of a closed-loop system to the open-loop frequency response and open-loop pole location. Thus, knowledge of the open-loop system's frequency response yields information about the stability of the closed-loop system. This concept is similar to the root locus, where we began with information about the open-loop system, its poles and zeros, and developed transient and stability information about the closed-loop system.

Although the Nyquist criterion will yield stability information at first, we will extend the concept to transient response and steady-state errors. Thus, frequency response techniques are an alternate approach to the root locus.

Derivation of the Nyquist Criterion

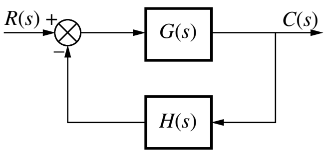

FIGURE 10.20 Closed-loop control system

Consider the system of Figure 10.20. The Nyquist criterion can tell us how many closed-loop poles are in the right half-plane. Before deriving the criterion, let us establish four important concepts that will be used during the derivation: (1) the relationship between the poles of $1 + G(s)H(s)$ and the poles of $G(s)H(s)$; (2) the relationship between the zeros of $1 + G(s)H(s)$ and the poles of the closed-loop transfer function, $T(s)$; (3) the concept of *mapping* points; and (4) the concept of mapping *contours*.

Letting

$$G(s) = \frac{N_G}{D_G} \tag{10.37a}$$

$$H(s) = \frac{N_H}{D_H} \tag{10.37b}$$

we find

$$G(s)H(s) = \frac{N_G N_H}{D_G D_H} \tag{10.38a}$$

$$1 + G(s)H(s) = 1 + \frac{N_G N_H}{D_G D_H} = \frac{D_G D_H + N_G N_H}{D_G D_H} \tag{10.38b}$$

$$T(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{N_G D_H}{D_G D_H + N_G N_H} \tag{10.38c}$$

From Eqs. (10.38), we conclude that (1) *the poles of* $1 + G(s)H(s)$ *are the same as the poles of* $G(s)H(s)$*, the open-loop system, and* (2) *the zeros of* $1 + G(s)H(s)$*are the same as the poles of* $T(s)$*, the closed-loop system.*

Next, let us define the term *mapping*. If we take a complex number on the s-plane and substitute it into a function, $F(s)$, another complex number results. This process is called *mapping*. For example, substituting $s = 4 + j3$ into the function $(s^2 + 2s + 1)$ yields $16 + j30$. We say that $4 + j3$ maps into $16 + j30$ through the function $(s^2 + 2s + 1)$.

Finally, we discuss the concept of mapping *contours*. Consider the collection of points, called a *contour*, shown in Figure 10.21 as contour A. Also, assume that

$$F(s) = \frac{(s - z_1)(s - z_2)\ldots}{(s - p_1)(s - p_2)\ldots} \tag{10.39}$$

Contour A can be mapped through $F(s)$ into contour B by substituting each point of contour A into the function $F(s)$ and plotting the resulting complex numbers. For example, point Q in Figure 10.21 maps into point Q' through the function $F(s)$.

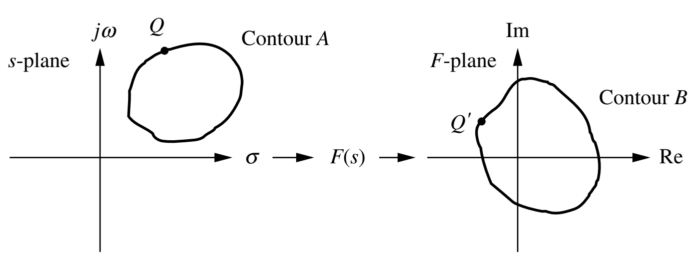

FIGURE 10.21 Mapping contour A through function $F(s)$ to contour B

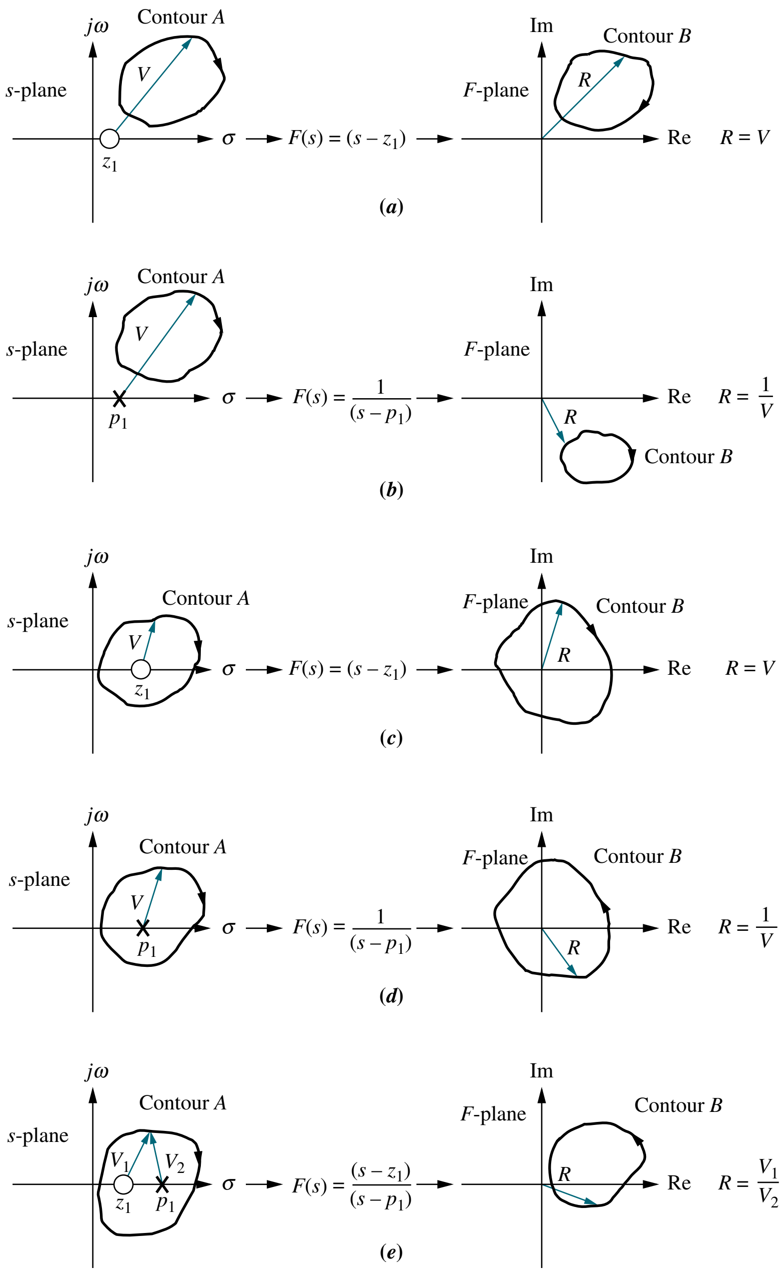

FIGURE 10.22 Examples of contour mapping

The vector approach to performing the calculation, covered in Section 8.1, can be used as an alternative. Some examples of contour mapping are shown in Figure 10.22 for some simple $F(s)$. The mapping of each point is defined by complex arithmetic, where the resulting complex number, R, is evaluated from the complex numbers represented by V, as shown in the last column of Figure 10.22. You should verify that if we assume a clockwise direction for mapping the points on contour A, then contour B maps in a clockwise direction if $F(s)$ in Figure 10.22 has just zeros or has just poles that are not encircled by the contour. The contour B maps in a counterclockwise direction if $F(s)$ has just poles that are encircled by the contour. Also, you should verify that if the pole or zero of $F(s)$ is enclosed by contour A, the

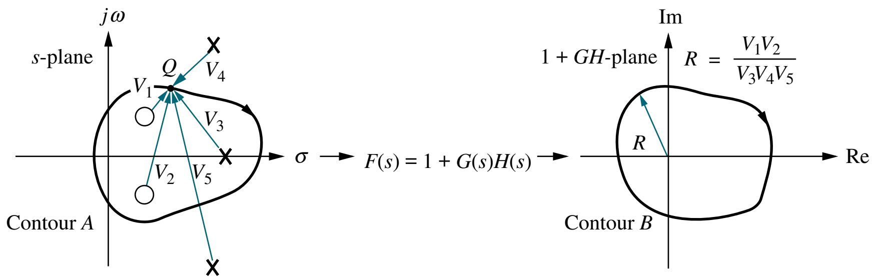

FIGURE 10.23 Vector representation of mapping

mapping encircles the origin. In the last case of Figure 10.22, the pole and zero rotation cancel, and the mapping does not encircle the origin.

Let us now begin the derivation of the Nyquist criterion for stability. We show that a unique relationship exists between the number of poles of $F(s)$ contained inside contour A, the number of zeros of $F(s)$ contained inside contour A, and the number of counterclockwise encirclements of the origin for the mapping of contour B. We then show how this interrelationship can be used to determine the stability of closed-loop systems. This method of determining stability is called the *Nyquist criterion*.

Let us first assume that $F(s) = 1 + G(s)H(s)$, with the picture of the poles and zeros of $1 + G(s)H(s)$ as shown in Figure 10.23 near contour A. Hence, $R = (V_1V_2)/(V_3V_4V_5)$. As each point Q of the contour A is substituted into $1 + G(s)H(s)$, a mapped point results on contour B. Assuming that $F(s) = 1 + G(s)H(s)$ has two zeros and three poles, each parenthetical term of Eq. (10.39) is a vector in Figure 10.23. As we move around contour A in a clockwise direction, each vector of Eq. (10.39) that lies inside contour A will appear to undergo a complete rotation, or a change in angle of 360°. On the other hand, each vector drawn from the poles and zeros of $1 + G(s)H(s)$ that exist outside contour A will appear to oscillate and return to its previous position, undergoing a net angular change of 0°.

Each pole or zero factor of $1 + G(s)H(s)$ whose vector undergoes a complete rotation around contour A must yield a change of 360° in the resultant, R, or a complete rotation of the mapping of contour B. If we move in a clockwise direction along contour A, each zero inside contour A yields a rotation in the clockwise direction, while each pole inside contour A yields a rotation in the counterclockwise direction since poles are in the denominator of Eq. (10.39).

Thus, $N = P - Z$, where N equals the number of counterclockwise rotations of contour B about the origin; P equals the number of poles of $1 + G(s)H(s)$ inside contour A, and Z equals the number of zeros of $1 + G(s)H(s)$ inside contour A.

Since the poles shown in Figure 10.23 are poles of $1 + G(s)H(s)$, we know from Eqs. (10.38) that they are also the poles of $G(s)H(s)$ and are known. But since *the zeros shown in Figure 10.23 are the zeros of* $1 + G(s)H(s)$, we know from Eqs. (10.38) that *they are also the poles of the closed-loop system and are not known*. Thus, P equals the number of enclosed open-loop poles, and Z equals the number of enclosed closed-loop poles. Hence, $N = P - Z$, or alternately, $Z = P - N$, tells us that the number of closed-loop poles inside the contour (which is the same as the zeros inside the contour) equals the number of open-loop poles of $G(s)H(s)$ inside the contour minus the number of counterclockwise rotations of the mapping about the origin.

If we extend the contour to include the entire right half-plane, as shown in Figure 10.24, we can count the number of right–half-plane, closed-loop poles inside contour A and determine a system's stability. Since we can count the number of open-loop poles, P, inside the contour, which are the same as the right–half-plane poles of $G(s)H(s)$, the only problem remaining is how to obtain the mapping and find N.

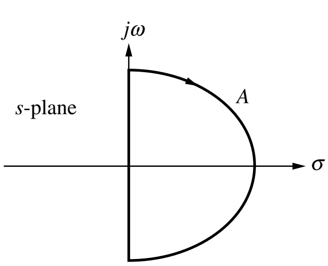

FIGURE 10.24 Contour enclosing right half-plane to determine stability

Since all of the poles and zeros of $G(s)H(s)$ are known, what if we map through $G(s)H(s)$ instead of $1 + G(s)H(s)$? The resulting contour is the same as a mapping through $1 + G(s)H(s)$, except that it is translated one unit to the left; thus, we count rotations about -1 instead of rotations about the origin. Hence, the final statement of the Nyquist stability criterion is as follows:

If a contour, A, that encircles the entire right half-plane is mapped through $G(s)H(s)$, then *the number of closed-loop poles, Z, in the right half-plane equals the* number of open-loop poles, P, that are in the right half-plane minus the number of counterclockwise revolutions, N, around -1 of the mapping; that is, $Z = P - N$. The mapping is called the *Nyquist diagram*, or *Nyquist plot, of* $G(s)H(s)$.

We can now see why this method is classified as a frequency response technique. Around contour A in Figure 10.24, the mapping of the points on the $j\omega$-axis through the function $G(s)H(s)$ is the same as substituting $s = j\omega$ into $G(s)H(s)$ to form the frequency response function $G(j\omega)H(j\omega)$. We are thus finding the frequency response of $G(s)H(s)$ over that part of contour A on the positive $j\omega$-axis. In other words, part of the Nyquist diagram is the polar plot of the frequency response of $G(s)H(s)$.

Applying the Nyquist Criterion to Determine Stability

Before describing how to sketch a Nyquist diagram, let us look at some typical examples that use the Nyquist criterion to determine the stability of a system. These examples give us a perspective prior to engaging in the details of mapping. Figure 10.25(a) shows a contour A that does not enclose closed-loop poles, that is, the zeros of $1 + G(s)H(s)$. The contour thus maps through $G(s)H(s)$ into a Nyquist diagram that does not encircle -1. Hence, $P = 0$, $N = 0$, and $Z = P - N = 0$. Since Z is the number of closed-loop poles inside contour A, which encircles the right half-plane, this system has no right–half-plane poles and is stable.

On the other hand, Figure 10.25(b) shows a contour A that, while it does not enclose open-loop poles, does generate two clockwise encirclements of -1. Thus, $P = 0$, $N = -2$, and the system is unstable; it has two closed-loop poles in the right half-plane since $Z = P - N = 2$. The two closed-loop poles are shown inside contour

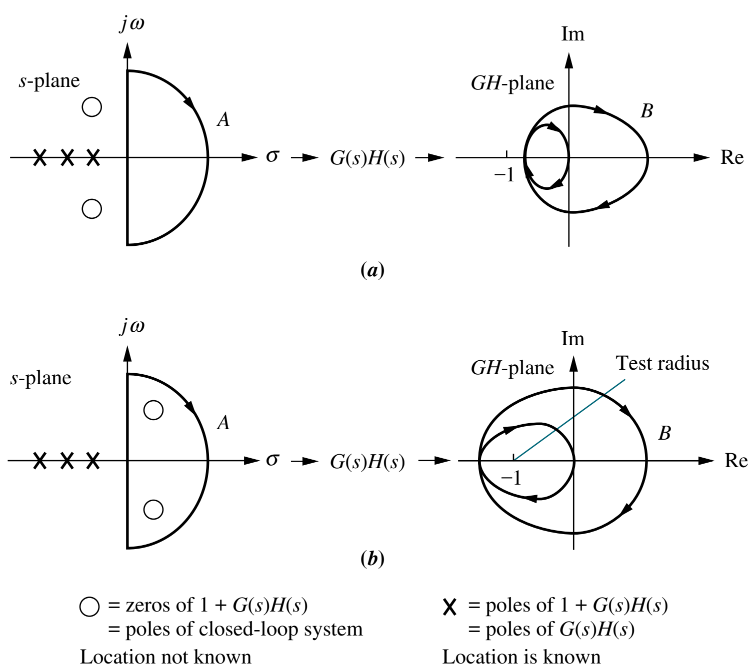

FIGURE 10.25 Mapping examples: **a.** Contour does not enclose closed-loop poles; **b.** contour does enclose closed-loop poles

A in Figure 10.25(b) as zeros of $1 + G(s)H(s)$. You should keep in mind that the existence of these poles is not known a priori.

In this example, notice that clockwise encirclements imply a negative value for N. The number of encirclements can be determined by drawing a test radius from -1 in any convenient direction and counting the number of times the Nyquist diagram crosses the test radius. Counterclockwise crossings are positive, and clockwise crossings are negative. For example, in Figure 10.25(b), contour B crosses the test radius twice in a clockwise direction. Hence, there are -2 encirclements of the point -1.

Before applying the Nyquist criterion to other examples in order to determine a system's stability, we must first gain experience in sketching Nyquist diagrams. The next section covers the development of this skill.

10.4 Sketching the Nyquist Diagram

The contour that encloses the right half-plane can be mapped through the function $G(s)H(s)$ by substituting points along the contour into $G(s)H(s)$. The points along the positive extension of the imaginary axis yield the polar frequency response of $G(s)H(s)$. Approximations can be made to $G(s)H(s)$ for points around the infinite semicircle by assuming that the vectors originate at the origin. Thus, their length is infinite, and their angles are easily evaluated.

However, most of the time a simple sketch of the Nyquist diagram is all that is needed. A sketch can be obtained rapidly by looking at the vectors of $G(s)H(s)$ and their motion along the contour. In the examples that follow, we stress this rapid method for sketching the Nyquist diagram. However, the examples also include analytical expressions for $G(s)H(s)$ for each section of the contour to aid you in determining the shape of the Nyquist diagram.

Example 10.4

Sketching a Nyquist Diagram

PROBLEM: Speed controls find wide application throughout industry and the home. Figure 10.26(a) shows one application: output frequency control of electrical

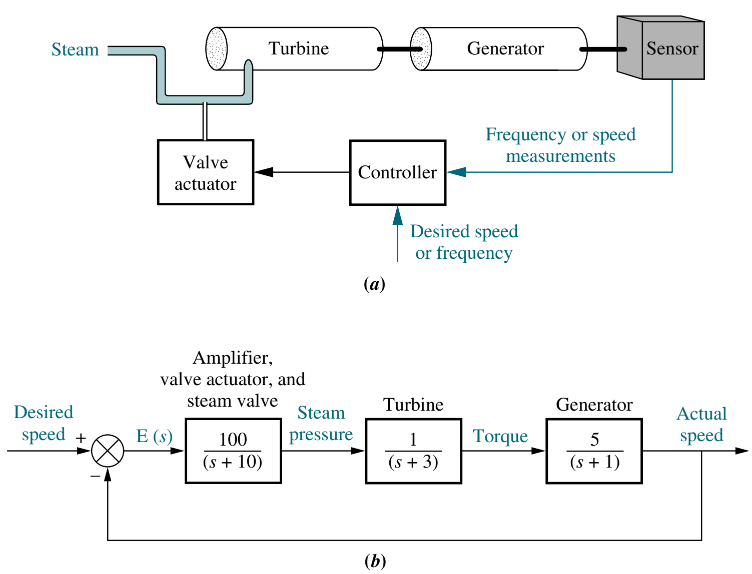

FIGURE 10.26
a. Turbine and generator; **b.** block diagram of speed control system for Example 10.4

power from a turbine and generator pair. By regulating the speed, the control system ensures that the generated frequency remains within tolerance. Deviations from the desired speed are sensed, and a steam valve is changed to compensate for the speed error. The system block diagram is shown in Figure 10.26(b). Sketch the Nyquist diagram for the system of Figure 10.26.

SOLUTION: Conceptually, the Nyquist diagram is plotted by substituting the points of the contour shown in Figure 10.27(a) into $G(s) = 500/[(s+1)(s+3)(s+10)]$. This process is equivalent to performing complex arithmetic using the vectors of $G(s)$ drawn to the points of the contour as shown in Figure 10.27(a) and (b). Each pole and zero term of $G(s)$ shown in Figure 10.26(b) is a vector in Figure 10.27(a) and (b). The resultant vector, R, found at any point along the contour is in general the product of the zero vectors divided by the product of the pole vectors (see Figure 10.27(c)). Thus, the magnitude of the resultant is the product of the zero lengths divided by the product of the pole lengths, and the angle of the resultant is the sum of the zero angles minus the sum of the pole angles.

As we move in a clockwise direction around the contour from point A to point C in Figure 10.27(a), the resultant angle goes from 0° to $-3 \times 90° = -270°$, or from A' to C' in Figure 10.27(c). Since the angles emanate from poles in the denominator of $G(s)$, the rotation or increase in angle is really a decrease in angle

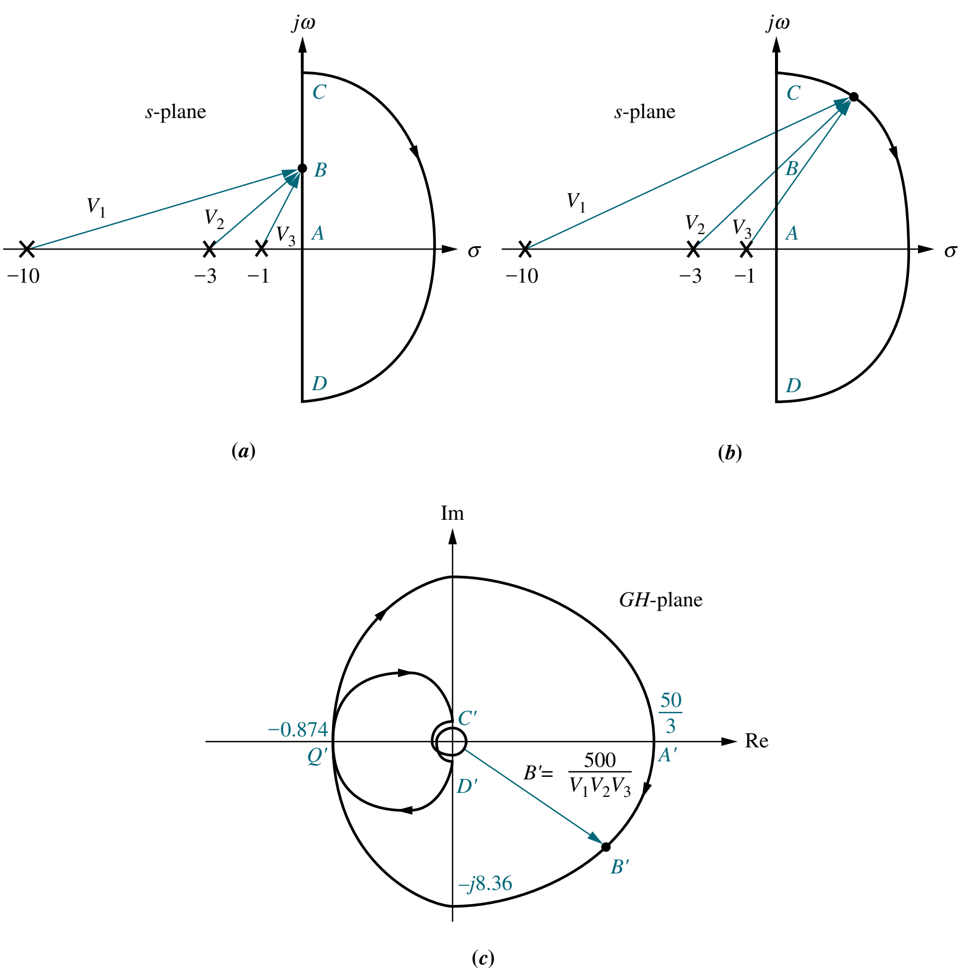

FIGURE 10.27 Vector evaluation of the Nyquist diagram for Example 10.4:
a. vectors on contour at low frequency;
b. vectors on contour around infinity;
c. Nyquist diagram

of the function $G(s)$; the poles gain 270° in a counterclockwise direction, which explains why the function loses 270°.

While the resultant moves from A' to C' in Figure 10.27(c), its magnitude changes as the product of the zero lengths divided by the product of the pole lengths. Thus, the resultant goes from a finite value at zero frequency (at point A of Figure 10.27(a), there are three finite pole lengths) to zero magnitude at infinite frequency at point C (at point C of Figure 10.27(a), there are three infinite pole lengths).

The mapping from point A to point C can also be explained analytically. From A to C the collection of points along the contour is imaginary. Hence, from A to C, $G(s) = G(j\omega)$, or from Figure 10.26(b),

$$G(j\omega) = \left.\frac{500}{(s+1)(s+3)(s+10)}\right|_{s\rightarrow j\omega} = \frac{500}{(-14\omega^2+30)+j(43\omega-\omega^3)} \tag{10.40}$$

Multiplying the numerator and denominator by the complex conjugate of the denominator, we obtain

$$G(j\omega) = 500\frac{(-14\omega^2+30)-j(43\omega-\omega^3)}{(-14\omega^2+30)^2+(43\omega-\omega^3)^2} \tag{10.41}$$

At zero frequency, $G(j\omega) = 500/30 = 50/3$. Thus, the Nyquist diagram starts at $50/3$ at an angle of 0°. As ω increases the real part remains positive, and the imaginary part remains negative. At $\omega = \sqrt{30/14}$, the real part becomes negative. At $\omega = \sqrt{43}$, the Nyquist diagram crosses the negative real axis since the imaginary term goes to zero. The real value at the axis crossing, point Q' in Figure 10.27(c), found by substituting into Eq. (10.41), is -0.874. Continuing toward $\omega = \infty$, the real part is negative, and the imaginary part is positive. At infinite frequency $G(j\omega) \approx 500j/\omega^3$, or approximately zero at 90°.

Around the infinite semicircle from point C to point D shown in Figure 10.27(b), the vectors rotate clockwise, each by 180°. Hence, the resultant undergoes a counterclockwise rotation of $3 \times 180°$, starting at point C' and ending at point D' of Figure 10.27(c). Analytically, we can see this by assuming that around the infinite semicircle, the vectors originate approximately at the origin and have infinite length. For any point on the s-plane, the value of $G(s)$ can be found by representing each complex number in polar form, as follows:

$$G(s) = \frac{500}{(R_{-1}e^{j\theta_{-1}})(R_{-3}e^{j\theta_{-3}})(R_{-10}e^{j\theta_{-10}})} \tag{10.42}$$

where R_{-i} is the magnitude of the complex number $(s+1)$, and θ_{-i} is the angle of the complex number $(s+i)$. Around the infinite semicircle, all R_{-i} are infinite, and we can use our assumption to approximate the angles as if the vectors originated at the origin. Thus, around the infinite semicircle,

$$G(s) = \frac{500}{\infty\angle(\theta_{-1}+\theta_{-3}+\theta_{-10})} = 0\angle-(\theta_{-1}+\theta_{-3}+\theta_{-10}) \tag{10.43}$$

At point C in Figure 10.27(b), the angles are all 90°. Hence, the resultant is $0\angle-270°$, shown as point C' in Figure 10.27(c). Similarly, at point D, $G(s) = 0\angle+270°$ and maps into point D'. You can select intermediate points to verify the spiral whose radius vector approaches zero at the origin, as shown in Figure 10.27(c).

The negative imaginary axis can be mapped by realizing that the real part of $G(j\omega)H(j\omega)$ is always an even function, whereas the imaginary part of $G(j\omega)H(j\omega)$ is an odd function. That is, the real part will not change sign when negative values of

ω are used, whereas the imaginary part will change sign. Thus, the mapping of the negative imaginary axis is a mirror image of the mapping of the positive imaginary axis. The mapping of the section of the contour from points D to A is drawn as a mirror image about the real axis of the mapping of points A to C.

In the previous example, there were no open-loop poles situated along the contour enclosing the right half-plane. If such poles exist, then a detour around the poles on the contour is required; otherwise, the mapping would go to infinity in an undetermined way, without angular information. Subsequently, a complete sketch of the Nyquist diagram could not be made, and the number of encirclements of -1 could not be found.

Let us assume a $G(s)H(s) = N(s)/sD(s)$ where $D(s)$ has imaginary roots. The s term in the denominator and the imaginary roots of $D(s)$ are poles of $G(s)H(s)$ that lie on the contour, as shown in Figure 10.28(a). To sketch the Nyquist diagram, the contour must detour around each open-loop pole lying on its path. The detour can be to the right of the pole, as shown in Figure 10.28(b), which makes it clear that each pole's vector rotates through $+180^\circ$ as we move around the contour near that pole. This knowledge of the angular rotation of the poles on the contour permits us to complete the Nyquist diagram. Of course, our detour must carry us only an infinitesimal distance into the right half-plane, or else some closed-loop, right–half-plane poles will be excluded in the count.

We can also detour to the left of the open-loop poles. In this case, each pole rotates through an angle of -180° as we detour around it. Again, the detour must be infinitesimally small, or else we might include some left–half-plane poles in the count. Let us look at an example.

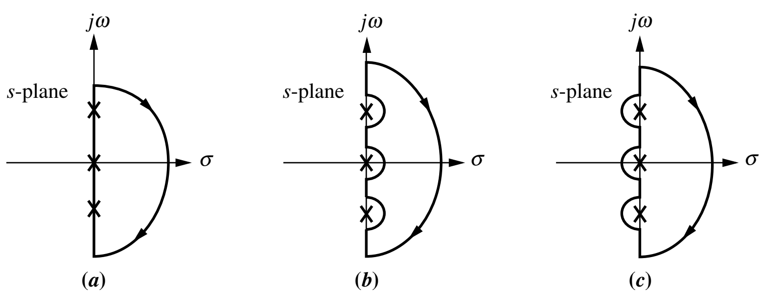

FIGURE 10.28 Detouring around open-loop poles: **a.** poles on contour; **b.** detour right; **c.** detour left

Example 10.5

Nyquist Diagram for Open-Loop Function with Poles on Contour

PROBLEM: Sketch the Nyquist diagram of the unity feedback system of Figure 10.10, where $G(s) = (s+2)/s^2$.

SOLUTION: The system's two poles at the origin are on the contour and must be bypassed, as shown in Figure 10.29(a). The mapping starts at point A and continues in a clockwise direction. Points A, B, C, D, E, and F of Figure 10.29(a) map respectively into points A', B', C', D', E', and F' of Figure 10.29(b).

At point A, the two open-loop poles at the origin contribute $2 \times 90^\circ = 180^\circ$, and the zero contributes 0°. The total angle at point A is thus -180°. Close to the origin, the function is infinite in magnitude because of the close proximity to the

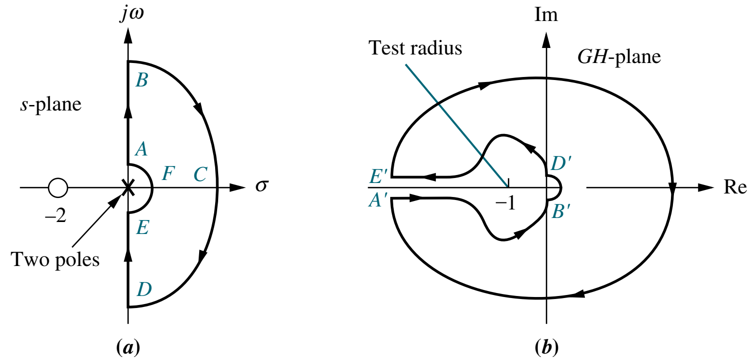

FIGURE 10.29 **a.** Contour for Example 10.5; **b.** Nyquist diagram for Example 10.5

two open-loop poles. Thus, point A maps into point A', located at infinity at an angle of $-180°$.

Moving from point A to point B along the contour yields a net change in angle of $+90°$ from the zero alone. The angles of the poles remain the same. Thus, the mapping changes by $+90°$ in the counterclockwise direction. The mapped vector goes from $-180°$ at A' to $-90°$ at B'. At the same time, the magnitude changes from infinity to zero since at point B there is one infinite length from the zero divided by two infinite lengths from the poles.

Alternately, the frequency response can be determined analytically from $G(j\omega) = (2 + j\omega)/(-\omega^2)$, considering ω going from 0 to ∞. At low frequencies, $G(j\omega) \approx 2/(-\omega^2)$, or $\infty\angle 180°$. At high frequencies, $G(j\omega) \approx j/(-\omega)$, or $0\angle -90°$. Also, the real and imaginary parts are always negative.

As we travel along the contour BCD, the function magnitude stays at zero (one infinite zero length divided by two infinite pole lengths). As the vectors move through BCD, the zero's vector and the two poles' vectors undergo changes of $-180°$ each. Thus, the mapped vector undergoes a net change of $+180°$, which is the angular change of the zero minus the sum of the angular changes of the poles $\{-180 - [2(-180)] = +180\}$. The mapping is shown as $B'\,C'\,D'$, where the resultant vector changes by $+180°$ with a magnitude of ϵ that approaches zero.

From the analytical point of view,

$$G(s) = \frac{R_{-2}\angle\theta_{-2}}{(R_0\angle\theta_0)(R_0\angle\theta_0)} \tag{10.44}$$

anywhere on the s-plane where $R_{-2}\angle\theta_{-2}$ is the vector from the zero at -2 to any point on the s-plane, and $R_0\angle\theta_0$ is the vector from a pole at the origin to any point on the s-plane. Around the infinite semicircle, all $R_{-i} = \infty$, and all angles can be approximated as if the vectors originated at the origin. Thus at point B, $G(s) = 0\angle -90°$ since all $\theta_{-i} = 90°$ in Eq. (10.44). At point C, all $R_{-i} = \infty$, and all $\theta_{-i} = 0°$ in Eq. (10.44). Thus, $G(s) = 0\angle 0°$. At point D, all $R_{-i} = \infty$, and all $\theta_{-i} = -90°$ in Eq. (10.44). Thus, $G(s) = 0\angle 90°$.

The mapping of the section of the contour from D to E is a mirror image of the mapping of A to B. The result is D' to E'.

Finally, over the section EFA, the resultant magnitude approaches infinity. The angle of the zero does not change, but each pole changes by $+180°$. This change yields a change in the function of $-2 \times 180° = -360°$. Thus, the mapping from E' to A' is shown as infinite in length and rotating $-360°$. Analytically, we can use Eq. (10.44) for the points along the contour EFA. At E,

$G(s) = (2\angle 0°)/[(\epsilon\angle -90°)(\epsilon\angle -90°)] = \infty\angle 180°$. At F, $G(s) = (2\angle 0°)/[(\epsilon\angle 0°)(\epsilon\angle 0°)] = \infty\angle 0°$. At A, $G(s) = (2\angle 0°)/[(\epsilon\angle 90°)(\epsilon\angle 90°)] = \infty\angle -180°$.

The Nyquist diagram is now complete, and a test radius drawn from -1 in Figure 10.29(b) shows one counterclockwise revolution, and one clockwise revolution, yielding zero encirclements.

MATLAB

ML

`Students who are using MATLAB should now run ch10p2 in Appendix B. You will learn how to use MATLAB to make a Nyquist plot and list the points on the plot. You will also learn how to specify a range for frequency. This exercise solves Example 10.5 using MATLAB.`

Skill-Assessment Exercise 10.3

PROBLEM: Sketch the Nyquist diagram for the system shown in Figure 10.10 where

$$G(s) = \frac{1}{(s+2)(s+4)}$$

Compare your sketch with the polar plot in Skill-Assessment Exercise 10.1(c).

ANSWER: The complete solution is located at www.wiley.com/college/nise.

In this section, we learned how to sketch a Nyquist diagram. We saw how to calculate the value of the intersection of the Nyquist diagram with the negative real axis. This intersection is important in determining the number of encirclements of -1. Also, we showed how to sketch the Nyquist diagram when open-loop poles exist on the contour; this case required detours around the poles. In the next section, we apply the Nyquist criterion to determine the stability of feedback control systems.

10.5 Stability via the Nyquist Diagram

We now use the Nyquist diagram to determine a system's stability, using the simple equation $Z = P - N$. The values of P, the number of open-loop poles of $G(s)H(s)$ enclosed by the contour, and N, the number of encirclements the Nyquist diagram makes about -1, are used to determine Z, the number of right–half-plane poles of the closed-loop system.

If the closed-loop system has a variable gain in the loop, one question we would like to ask is, "For what range of gain is the system stable?" This question, previously answered by the root locus method and the Routh-Hurwitz criterion, is now answered via the Nyquist criterion. The general approach is to set the loop gain equal to unity and draw the Nyquist diagram. Since gain is simply a multiplying factor, the effect of the gain is to multiply the resultant by a constant anywhere along the Nyquist diagram.

For example, consider Figure 10.30, which summarizes the Nyquist approach for a system with variable gain, K. As the gain is varied, we can visualize the Nyquist diagram in Figure 10.30(c) expanding (increased gain) or shrinking (decreased gain) like a balloon. This motion could move the Nyquist diagram past the -1 point, changing the stability picture. For this system, since $P = 2$, the critical point must be encircled by the Nyquist diagram to yield $N = 2$ and a stable system. A reduction in

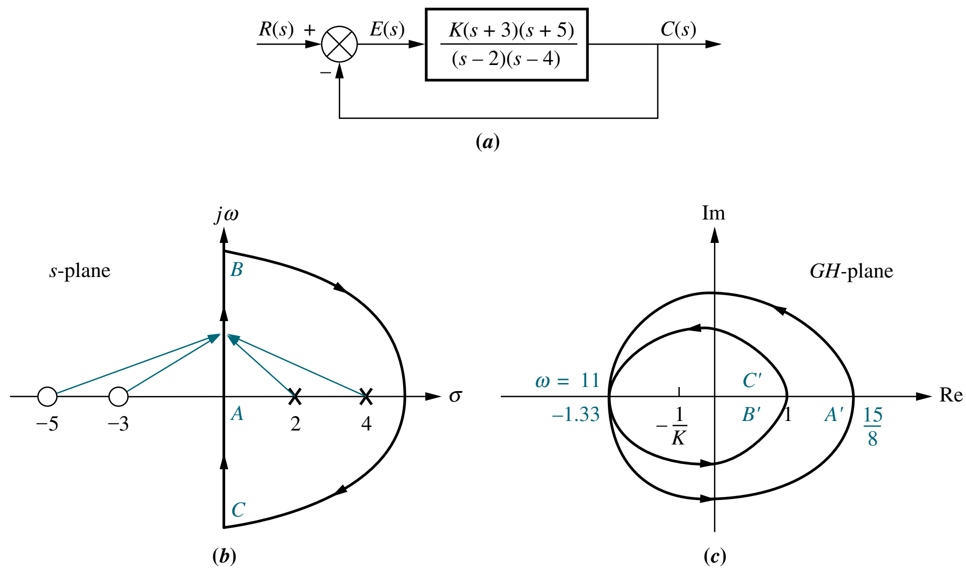

FIGURE 10.30 Demonstrating Nyquist stability: **a.** system; **b.** contour; **c.** Nyquist diagram

TryIt 10.2

Use MATLAB, the Control System Toolbox, and the following statements to plot the Nyquist diagram of the system shown in Figure 10.30(a).

```
G=zpk([-3,-5],...
 [2,4],1)
nyquist(G)
```

After the Nyquist diagram appears, click on the curve and drag to read the coordinates.

gain would place the critical point outside the Nyquist diagram where $N = 0$, yielding $Z = 2$, an unstable system.

From another perspective we can think of the Nyquist diagram as remaining stationary and the -1 point moving along the real axis. In order to do this, we set the gain to unity and position the critical point at $-1/K$ rather than -1. Thus, the critical point appears to move closer to the origin as K increases.

Finally, if the Nyquist diagram intersects the real axis at -1, then $G(j\omega)H(j\omega) = -1$. From root locus concepts, when $G(s)H(s) = -1$, the variable s is a closed-loop pole of the system. Thus, the frequency at which the Nyquist diagram intersects -1 is the same frequency at which the root locus crosses the $j\omega$-axis. Hence, the system is marginally stable if the Nyquist diagram intersects the real axis at -1.

In summary, then, if the open-loop system contains a variable gain, K, set $K = 1$ and sketch the Nyquist diagram. Consider the critical point to be at $-1/K$ rather than at -1. Adjust the value of K to yield stability, based upon the Nyquist criterion.

Example 10.6

Range of Gain for Stability via The Nyquist Criterion

PROBLEM: For the unity feedback system of Figure 10.10, where $G(s) = K/[s(s+3)(s+5)]$, find the range of gain, K, for stability, instability, and the value of gain for marginal stability. For marginal stability also find the frequency of oscillation. Use the Nyquist criterion.

SOLUTION: First set $K = 1$ and sketch the Nyquist diagram for the system, using the contour shown in Figure 10.31(a). For all points on the imaginary axis,

$$G(j\omega)H(j\omega) = \frac{K}{s(s+3)(s+5)}\bigg|_{\substack{K=1\\ s=j\omega}} = \frac{-8\omega^2 - j(15\omega - \omega^3)}{64\omega^4 + \omega^2(15-\omega^2)^2} \qquad (10.45)$$

At $\omega = 0$, $G(j\omega)H(j\omega) = -0.0356 - j\infty$.

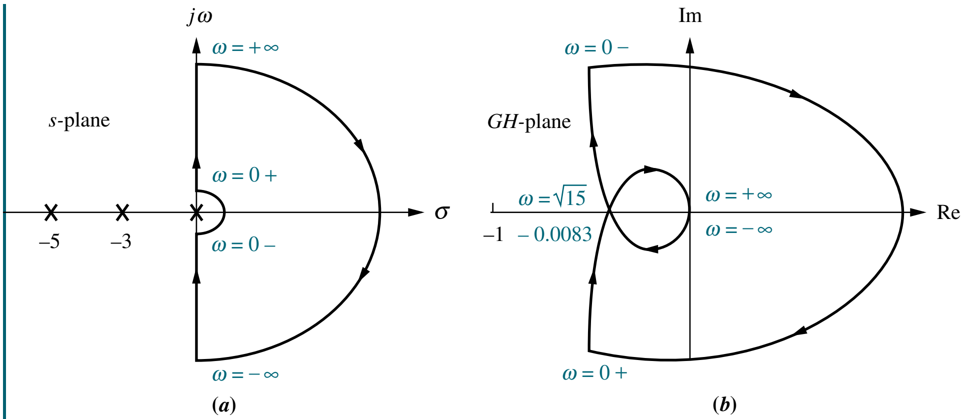

FIGURE 10.31
a. Contour for Example 10.6; **b.** Nyquist diagram

Next find the point where the Nyquist diagram intersects the negative real axis. Setting the imaginary part of Eq. (10.45) equal to zero, we find $\omega = \sqrt{15}$. Substituting this value of ω back into Eq. (10.45) yields the real part of -0.0083. Finally, at $\omega = \infty$, $G(j\omega)H(j\omega) = G(s)H(s)|_{s\to j\infty} = 1/(j\infty)^3 = 0\angle -270°$.

From the contour of Figure 10.31(a), $P = 0$; for stability N must then be equal to zero. From Figure 10.31(b), the system is stable if the critical point lies outside the contour ($N = 0$), so that $Z = P - N = 0$. Thus, K can be increased by $1/0.0083 = 120.5$ before the Nyquist diagram encircles -1. Hence, for stability, $K < 120.5$. For marginal stability $K = 120.5$. At this gain the Nyquist diagram intersects -1, and the frequency of oscillation is $\sqrt{15}$ rad/s.

Now that we have used the Nyquist diagram to determine stability, we can develop a simplified approach that uses only the mapping of the positive $j\omega$-axis.

Stability via Mapping Only the Positive $j\omega$-Axis

Once the stability of a system is determined by the Nyquist criterion, continued evaluation of the system can be simplified by using just the mapping of the positive $j\omega$-axis. This concept plays a major role in the next two sections, where we discuss stability margin and the implementation of the Nyquist criterion with Bode plots.

Consider the system shown in Figure 10.32, which is stable at low values of gain and unstable at high values of gain. Since the contour does not encircle open-loop

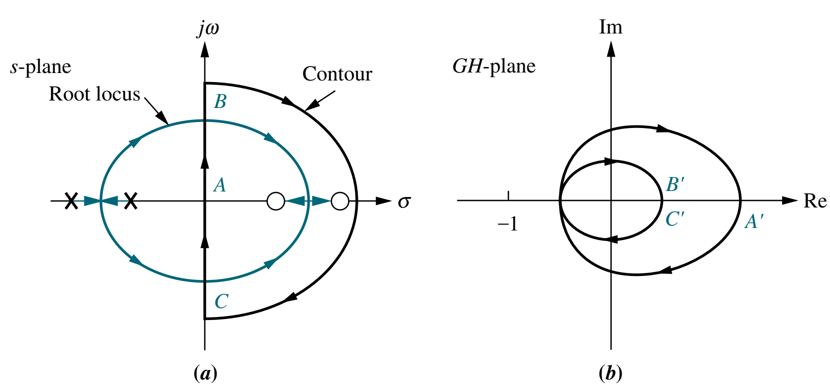

FIGURE 10.32
a. Contour and root locus of system that is stable for small gain and unstable for large gain; **b.** Nyquist diagram

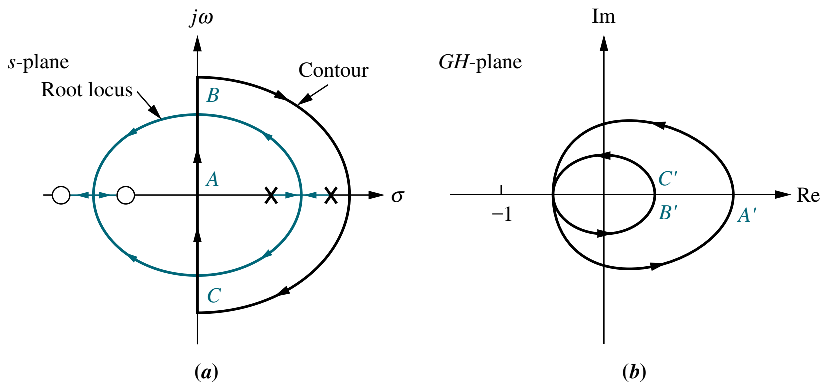

FIGURE 10.33 **a.** Contour and root locus of system that is unstable for small gain and stable for large gain; **b.** Nyquist diagram

poles, the Nyquist criterion tells us that we must have no encirclements of -1 for the system to be stable. We can see from the Nyquist diagram that the encirclements of the critical point can be determined from the mapping of the positive $j\omega$-axis alone. If the gain is small, the mapping will pass to the right of -1, and the system will be stable. If the gain is high, the mapping will pass to the left of -1, and the system will be unstable. Thus, this system is stable for the range of loop gain, K, that ensures that the *open-loop magnitude is less than unity at that frequency where the phase angle is* $180°$ (*or, equivalently*, $-180°$). This statement is thus an alternative to the Nyquist criterion for this system.

Now consider the system shown in Figure 10.33, which is unstable at low values of gain and stable at high values of gain. Since the contour encloses two open-loop poles, two counterclockwise encirclements of the critical point are required for stability. Thus, for this case the system is stable if the *open-loop magnitude is greater than unity at that frequency where the phase angle is* $180°$ (*or, equivalently*, $-180°$).

In summary, first determine stability from the Nyquist criterion and the Nyquist diagram. Next interpret the Nyquist criterion and determine whether the mapping of just the positive imaginary axis should have a gain of less than or greater than unity at $180°$. If the Nyquist diagram crosses $\pm180°$ at multiple frequencies, determine the interpretation from the Nyquist criterion.

Example 10.7

Stability Design via Mapping Positive $j\omega$-Axis

PROBLEM: Find the range of gain for stability and instability, and the gain for marginal stability, for the unity feedback system shown in Figure 10.10, where $G(s) = K/[(s^2 + 2s + 2)(s + 2)]$. For marginal stability find the radian frequency of oscillation. Use the Nyquist criterion and the mapping of only the positive imaginary axis.

SOLUTION: Since the open-loop poles are only in the left–half-plane, the Nyquist criterion tells us that we want no encirclements of -1 for stability. Hence, a gain less than unity at $\pm180°$ is required. Begin by letting $K = 1$ and draw the portion of the contour along the positive imaginary axis as shown in Figure 10.34(a). In

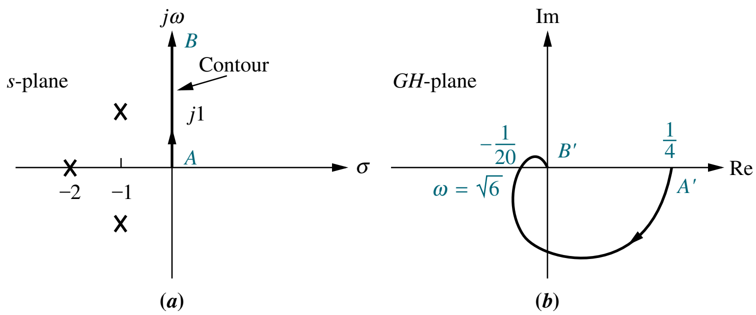

FIGURE 10.34 **a.** Portion of contour to be mapped for Example 10.7; **b.** Nyquist diagram of mapping of positive imaginary axis

Figure 10.34(*b*), the intersection with the negative real axis is found by letting $s = j\omega$ in $G(s)H(s)$, setting the imaginary part equal to zero to find the frequency, and then substituting the frequency into the real part of $G(j\omega)H(j\omega)$. Thus, for any point on the positive imaginary axis,

$$\begin{aligned} G(j\omega)H(j\omega) &= \left.\frac{1}{(s^2+2s+2)(s+2)}\right|_{s\to j\omega} \\ &= \frac{4(1-\omega^2) - j\omega(6-\omega^2)}{16(1-\omega^2)^2 + \omega^2(6-\omega^2)^2} \end{aligned} \tag{10.46}$$

Setting the imaginary part equal to zero, we find $\omega = \sqrt{6}$. Substituting this value back into Eq. (10.46) yields the real part, $-(1/20) = (1/20)\angle 180^\circ$.

This closed-loop system is stable if the magnitude of the frequency response is less than unity at 180°. Hence, the system is stable for $K < 20$, unstable for $K > 20$, and marginally stable for $K = 20$. When the system is marginally stable, the radian frequency of oscillation is $\sqrt{6}$.

Skill-Assessment Exercise 10.4

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PROBLEM: For the system shown in Figure 10.10, where

$$G(s) = \frac{K}{(s+2)(s+4)(s+6)}$$

do the following:

a. Plot the Nyquist diagram.

b. Use your Nyquist diagram to find the range of gain, K, for stability.

ANSWERS:

a. See the answer at www.wiley.com/college/nise.

b. Stable for $K < 480$

The complete solution is at www.wiley.com/college/nise.

10.6 Gain Margin and Phase Margin via the Nyquist Diagram

Now that we know how to sketch and interpret a Nyquist diagram to determine a closed-loop system's stability, let us extend our discussion to concepts that will eventually lead us to the design of transient response characteristics via frequency response techniques.

Using the Nyquist diagram, we define two quantitative measures of how stable a system is. These quantities are called *gain margin* and *phase margin*. Systems with greater gain and phase margins can withstand greater changes in system parameters before becoming unstable. In a sense, gain and phase margins can be qualitatively related to the root locus, in that systems whose poles are farther from the imaginary axis have a greater degree of stability.

In the last section, we discussed stability from the point of view of gain at 180° phase shift. This concept leads to the following definitions of gain margin and phase margin:

Gain margin, G_M. The gain margin is the change in open-loop gain, expressed in decibels (dB), required at 180° of phase shift to make the closed-loop system unstable.

Phase margin, Φ_M. The phase margin is the change in open-loop phase shift required at unity gain to make the closed-loop system unstable.

These two definitions are shown graphically on the Nyquist diagram in Figure 10.35.

Assume a system that is stable if there are no encirclements of -1. Using Figure 10.35, let us focus on the definition of gain margin. Here a gain difference between the Nyquist diagram's crossing of the real axis at $-1/a$ and the -1 critical point determines the proximity of the system to instability. Thus, if the gain of the system were multiplied by a units, the Nyquist diagram would intersect the critical point. We then say that the gain margin is a units, or, expressed in dB, $G_M = 20\log a$. Notice that the gain margin is the reciprocal of the real-axis crossing expressed in dB.

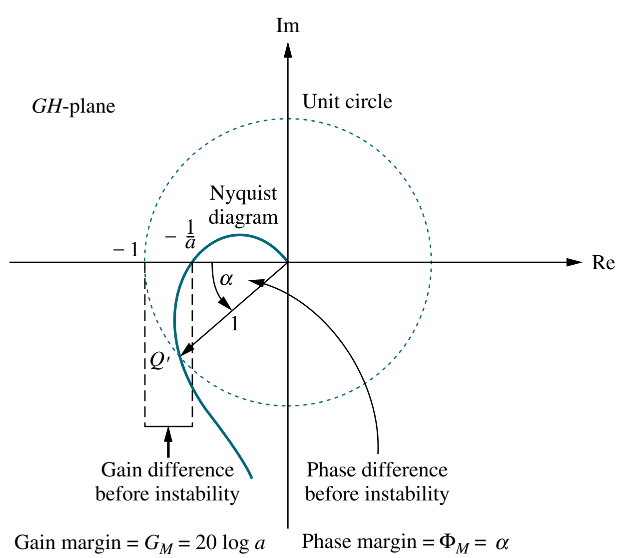

FIGURE 10.35 Nyquist diagram showing gain and phase margins

In Figure 10.35, we also see the phase margin graphically displayed. At point Q', where the gain is unity, α represents the system's proximity to instability. That is, at unity gain, if a phase shift of α degrees occurs, the system becomes unstable. Hence, the amount of phase margin is α. Later in the chapter, we show that phase margin can be related to the damping ratio. Thus, we will be able to relate frequency response characteristics to transient response characteristics as well as stability. We will also show that the calculations of gain and phase margins are more convenient if Bode plots are used rather than a Nyquist diagram, such as that shown in Figure 10.35.

For now let us look at an example that shows the calculation of the gain and phase margins.

Example 10.8

Finding Gain and Phase Margins

PROBLEM: Find the gain and phase margin for the system of Example 10.7 if $K = 6$.

SOLUTION: To find the gain margin, first find the frequency where the Nyquist diagram crosses the negative real axis. Finding $G(j\omega)H(j\omega)$, we have

$$\begin{aligned} G(j\omega)H(j\omega) &= \left.\frac{6}{(s^2+2s+2)(s+2)}\right|_{s\to j\omega} \\ &= \frac{6[4(1-\omega^2) - j\omega(6-\omega^2)]}{16(1-\omega^2)^2 + \omega^2(6-\omega^2)^2} \end{aligned} \tag{10.47}$$

The Nyquist diagram crosses the real axis at a frequency of $\sqrt{6}$ rad/s. The real part is calculated to be -0.3. Thus, the gain can be increased by $(1/0.3) = 3.33$ before the real part becomes -1. Hence, the gain margin is

$$G_M = 20 \log 3.33 = 10.45 \text{ dB} \tag{10.48}$$

To find the phase margin, find the frequency in Eq. (10.47) for which the magnitude is unity. As the problem stands, this calculation requires computational tools, such as a function solver or the program described in Appendix H.2. Later in the chapter we will simplify the process by using Bode plots. Eq. (10.47) has unity gain at a frequency of 1.253 rad/s. At this frequency, the phase angle is $-112.3°$. The difference between this angle and $-180°$ is $67.7°$, which is the phase margin.

MATLAB
ML

`Students who are using MATLAB should now run ch10p3 in Appendix B. You will learn how to use MATLAB to find gain margin, phase margin, zero dB frequency, and 180° frequency. This exercise solves Example 10.8 using MATLAB.`

Gui Tool
GUIT

`MATLAB's LTI Viewer, with the Nyquist diagram selected, is another method that may be used to find gain margin, phase margin, zero dB frequency, and 180° frequency. You are encouraged to study Appendix E, at www.wiley.com/college/nise, which contains a tutorial on the LTI Viewer as well as some examples. Example E.2 solves Example 10.8 using the LTI Viewer.`

Skill-Assessment Exercise 10.5

TryIt 10.3

Use MATLAB, the Control System Toolbox, and the following statements to find the gain and phase margins of $G(s)H(s) = 100/[(s+2)(s+4)(s+6)]$ using the Nyquist diagram.

```
G=zpk([ ],[ -2,-4,-6],100)
nyquist(G)
```

After the Nyquist diagram appears:

1. Right-click in the graph area.
2. Select **Characteristics**.
3. Select **All Stability Margins**.
4. Let the mouse rest on the margin points to read the gain and phase margins.

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Control Solutions

PROBLEM: Find the gain margin and the 180° frequency for the problem in Skill-Assessment Exercise 10.4 if $K = 100$.

ANSWERS: Gain margin = 13.62 dB; 180° frequency = 6.63 rad/s

The complete solution is at www.wiley.com/college/nise.

In this section, we defined gain margin and phase margin and calculated them via the Nyquist diagram. In the next section, we show how to use Bode diagrams to implement the stability calculations performed in Sections 10.5 and 10.6 using the Nyquist diagram. We will see that the Bode plots reduce the time and simplify the calculations required to obtain results.

10.7 Stability, Gain Margin, and Phase Margin via Bode Plots

In this section, we determine stability, gain and phase margins, and the range of gain required for stability. All of these topics were covered previously in this chapter, using Nyquist diagrams as the tool. Now we use Bode plots to determine these characteristics. Bode plots are subsets of the complete Nyquist diagram but in another form. They are a viable alternative to Nyquist plots, since they are easily drawn without the aid of the computational devices or long calculations required for the Nyquist diagram and root locus. You should remember that all calculations applied to stability were derived from and based upon the Nyquist stability criterion. The Bode plots are an alternate way of visualizing and implementing the theoretical concepts.

Determining Stability

Let us look at an example and determine the stability of a system, implementing the Nyquist stability criterion using Bode plots. We will draw a Bode log-magnitude plot and then determine the value of gain that ensures that the magnitude is less than 0 dB (unity gain) at that frequency where the phase is ±180°.

Example 10.9

Range of Gain for Stability via Bode Plots

PROBLEM: Use Bode plots to determine the range of K within which the unity feedback system shown in Figure 10.10 is stable. Let $G(s) = K/[(s+2)(s+4)(s+5)]$.

SOLUTION: Since this system has all of its open-loop poles in the left–half-plane, the open-loop system is stable. Hence, from the discussion of Section 10.5, the closed-loop system will be stable if the frequency response has a gain less than unity when the phase is 180°.

Begin by sketching the Bode magnitude and phase diagrams shown in Figure 10.36. In Section 10.2, we summed normalized plots of each factor of $G(s)$ to create the Bode plot. We saw that at each break frequency, the slope of the resultant Bode plot changed by an amount equal to the new slope that was added. Table 10.6 demonstrates this observation. In this example, we use this fact to draw the Bode plots faster by avoiding the sketching of the response of each term.

The low-frequency gain of $G(s)H(s)$ is found by setting s to zero. Thus, the Bode magnitude plot starts at $K/40$. For convenience, let $K = 40$ so that the log-magnitude plot starts at 0 dB. At each break frequency, 2, 4, and 5, a 20 dB/decade increase in negative slope is drawn, yielding the log-magnitude plot shown in Figure 10.36.

The phase diagram begins at 0° until a decade below the first break frequency of 2 rad/s. At 0.2 rad/s the curve decreases at a rate of $-45°$/decade, decreasing an additional 45°/decade at each subsequent frequency (0.4 and 0.5 rad/s) a decade below each break. At a decade above each break frequency, the slopes are reduced by 45°/decade at each frequency.

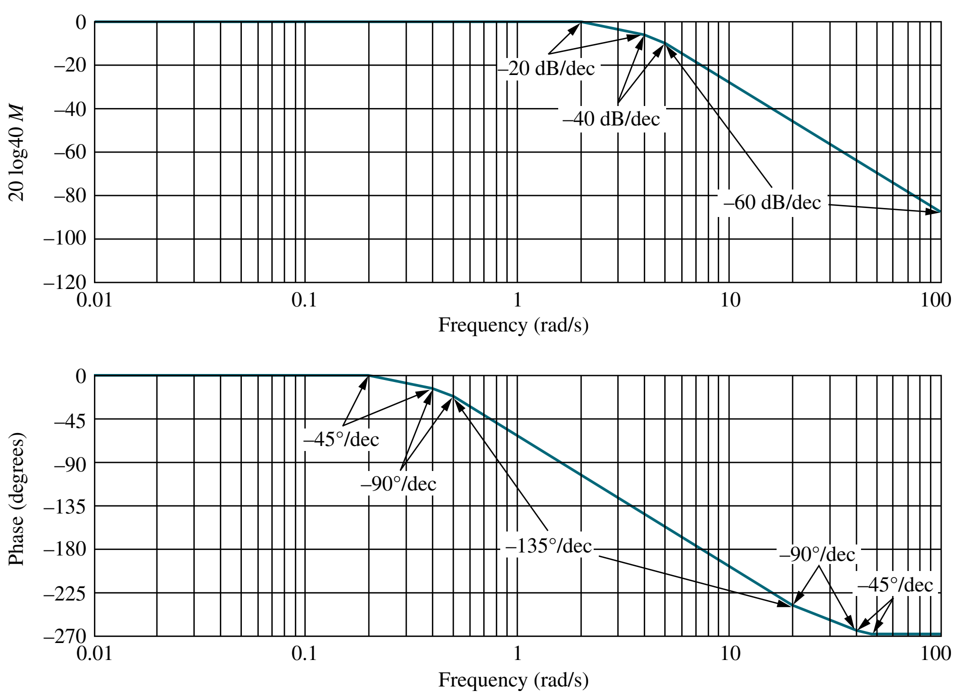

FIGURE 10.36 Bode log-magnitude and phase diagrams for the system of Example 10.9

The Nyquist criterion for this example tells us that we want zero encirclements of -1 for stability. Thus, we recognize that the Bode log-magnitude plot must be less than unity when the Bode phase plot is 180°. Accordingly, we see that at a frequency of 7 rad/s, when the phase plot is $-180°$, the magnitude plot is -20 dB. Therefore, an increase in gain of $+20$ dB is possible before the system becomes unstable. Since the gain plot was scaled for a gain of 40, $+20$ dB (a gain of 10) represents the required increase in gain above 40. Hence, the gain for instability is $40 \times 10 = 400$. The final result is $0 < K < 400$ for stability.

This result, obtained by approximating the frequency response by Bode asymptotes, can be compared to the result obtained from the actual frequency response, which yields a gain of 378 at a frequency of 6.16 rad/s.

MATLAB
ML

Students who are using MATLAB should now run `ch10p4` in Appendix B. You will learn how to use MATLAB to find the range of gain for stability via frequency response methods. This exercise solves Example 10.9 using MATLAB.

Evaluating Gain and Phase Margins

Next we show how to evaluate the gain and phase margins by using Bode plots (Figure 10.37). The gain margin is found by using the phase plot to find the frequency, ω_{G_M}, where the phase angle is 180°. At this frequency, we look at the magnitude plot to determine the gain margin, G_M, which is the gain required to raise the magnitude curve to 0 dB. To illustrate, in the previous example with $K = 40$, the gain margin was found to be 20 dB.

The phase margin is found by using the magnitude curve to find the frequency, ω_{Φ_M}, where the gain is 0 dB. On the phase curve at that frequency, the phase margin, ϕ_M, is the difference between the phase value and 180°.

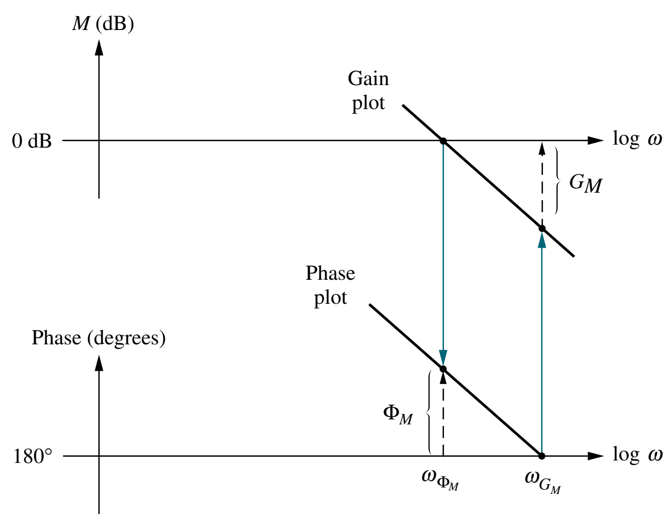

FIGURE 10.37 Gain and phase margins on the Bode diagrams

Example 10.10

Gain and Phase Margins from Bode Plots

PROBLEM: If $K = 200$ in the system of Example 10.9, find the gain margin and the phase margin.

SOLUTION: The Bode plot in Figure 10.36 is scaled to a gain of 40. If $K = 200$ (five times as great), the magnitude plot would be $20 \log 5 = 13.98$ dB higher.

To find the gain margin, look at the phase plot and find the frequency where the phase is 180°. At this frequency, determine from the magnitude plot how much the gain can be increased before reaching 0 dB. In Figure 10.36, the phase angle is 180° at approximately 7 rad/s. On the magnitude plot, the gain is $-20 + 13.98 = -6.02$ dB. Thus, the gain margin is 6.02 dB.

To find the phase margin, we look on the magnitude plot for the frequency where the gain is 0 dB. At this frequency, we look on the phase plot to find the difference between the phase and 180°. This difference is the phase margin. Again, remembering that the magnitude plot of Figure 10.36 is 13.98 dB lower than the actual plot, the 0 dB crossing (−13.98 dB for the normalized plot shown in Figure 10.36) occurs at 5.5 rad/s. At this frequency the phase angle is −165°. Thus, the phase margin is $-165° - (-180°) = 15°$.

Gui Tool
GUIT

`MATLAB's LTI Viewer, with Bode plots selected, is another method that may be used to find gain margin, phase margin, zero dB frequency, and 180° frequency. You are encouraged to study Appendix E at www.wiley.com/college/nise, which contains a tutorial on the LTI Viewer as well as some examples. Example E.3 solves Example 10.10 using the LTI Viewer.`

Skill-Assessment Exercise 10.6

PROBLEM: For the system shown in Figure 10.10, where

$$G(s) = \frac{K}{(s+5)(s+20)(s+50)}$$

do the following:

a. Draw the Bode log-magnitude and phase plots.

b. Find the range of K for stability from your Bode plots.

c. Evaluate gain margin, phase margin, zero dB frequency, and 180° frequency from your Bode plots for $K = 10,000$.

ANSWERS:

a. See the answer at www.wiley.com/college/nise.

b. $K < 96,270$

c. Gain margin = 19.67 dB, phase margin = 92.9°, zero dB frequency = 7.74 rad/s, and 180° frequency = 36.7 rad/s

The complete solution is at www.wiley.com/college/nise.

TryIt 10.4

Use MATLAB, the Control System Toolbox, and the following statements to solve Skill-Assessment Exercise 10.6(c) using Bode plots.

```
G=zpk([],...
 [-5,-20,-50],10000)
bode(G)
grid on
```

After the Bode plot appears:

1. Right-click in the graph area.
2. Select **Characteristics**.
3. Select **All Stability Margins**.
4. Let the mouse rest on the margin points to read the gain and phase margins.

We have seen that the open-loop frequency response curves can be used not only to determine whether a system is stable but to calculate the range of loop gain that will ensure stability. We have also seen how to calculate the gain margin and the phase margin from the Bode diagrams.

Is it then possible to parallel the root locus technique and analyze and design systems for transient response using frequency response methods? We will begin to explore the answer in the next section.

10.8 Relation Between Closed-Loop Transient and Closed-Loop Frequency Responses

Damping Ratio and Closed-Loop Frequency Response

In this section, we will show that a relationship exists between a system's transient response and its closed-loop frequency response. In particular, consider the second-order feedback control system of Figure 10.38, which we have been using since Chapter 4, where we derived relationships between the closed-loop transient response and the poles of the closed-loop transfer function,

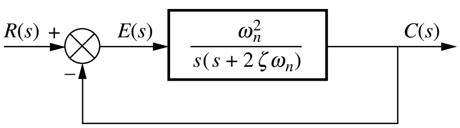

FIGURE 10.38 Second-order closed-loop system

$$\frac{C(s)}{R(s)} = T(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{10.49}$$

We now derive relationships between the transient response of Eq. (10.49) and characteristics of its frequency response. We define these characteristics and relate them to damping ratio, natural frequency, settling time, peak time, and rise time. In Section 10.10, we will show how to use the frequency response of the open-loop transfer function

$$G(s) = \frac{\omega_n^2}{s(s + 2\zeta\omega_n)} \tag{10.50}$$

shown in Figure 10.38, to obtain the same transient response characteristics.

Let us now find the frequency response of Eq. (10.49), define characteristics of this response, and relate these characteristics to the transient response. Substituting $s = j\omega$ into Eq. (10.49), we evaluate the magnitude of the closed-loop frequency response as

$$M = |T(j\omega)| = \frac{\omega_n^2}{\sqrt{(\omega_n^2 - \omega^2)^2 + 4\zeta^2\omega_n^2\omega^2}} \tag{10.51}$$

A representative sketch of the log plot of Eq. (10.51) is shown in Figure 10.39.

We now show that a relationship exists between the peak value of the closed-loop magnitude response and the damping ratio. Squaring Eq. (10.51), differentiating with respect to ω^2, and setting the derivative equal to zero yields the maximum value of M, M_p, where

$$\boxed{M_p = \frac{1}{2\zeta\sqrt{1 - \zeta^2}}} \tag{10.52}$$

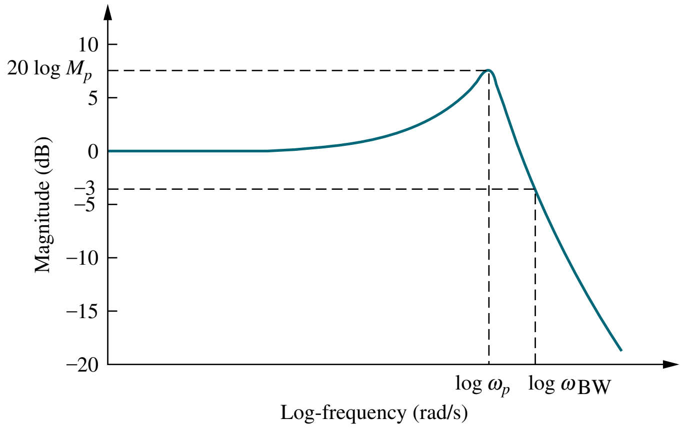

FIGURE 10.39 Representative log-magnitude plot of Eq. (10.51)

at a frequency, ω_p, of

$$\boxed{\omega_p = \omega_n\sqrt{1-2\zeta^2}} \tag{10.53}$$

Since ζ is related to percent overshoot, we can plot M_p vs. percent overshoot. The result is shown in Figure 10.40.

Equation (10.52) shows that the maximum magnitude on the frequency response curve is directly related to the damping ratio and, hence, the percent overshoot. Also notice from Eq. (10.53) that the peak frequency, ω_p, is not the natural frequency. However, for low values of damping ratio, we can assume that the peak occurs at the natural frequency. Finally, notice that there will not be a peak at frequencies above zero if $\zeta > 0.707$. This limiting value of ζ for peaking on the magnitude response curve should not be confused with overshoot on the step response, where there is overshoot for $0 < \zeta < 1$.

Response Speed and Closed-Loop Frequency Response

Another relationship between the frequency response and time response is between the speed of the time response (as measured by settling time, peak time, and rise time) and the *bandwidth* of the closed-loop frequency response, which is defined here as the frequency, ω_{BW}, at which the magnitude response curve is 3 dB down from its value at zero frequency (see Figure 10.39).

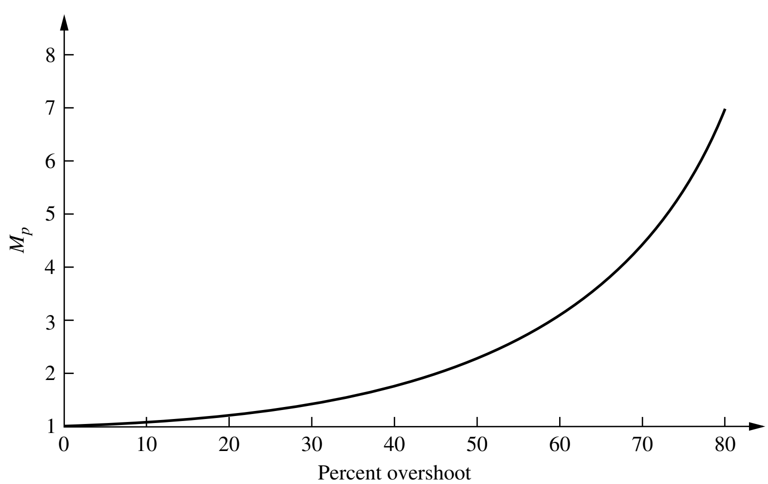

FIGURE 10.40 Closed-loop frequency response peak vs. percent overshoot for a two-pole system

The bandwidth of a two-pole system can be found by finding that frequency for which $M = 1/\sqrt{2}$ (that is, -3 dB) in Eq.(10.51). The derivation is left as an exercise for the student. The result is

$$\boxed{\omega_{BW} = \omega_n\sqrt{(1-2\zeta^2)+\sqrt{4\zeta^4-4\zeta^2+2}}} \tag{10.54}$$

To relate ω_{BW} to settling time, we substitute $\omega_n = 4/T_s\zeta$ into Eq. (10.54) and obtain

$$\boxed{\omega_{BW} = \frac{4}{T_s\zeta}\sqrt{(1-2\zeta^2)+\sqrt{4\zeta^4-4\zeta^2+2}}} \tag{10.55}$$

Similarly, since, $\omega_n = \pi/(T_p\sqrt{1-\zeta^2})$,

$$\boxed{\omega_{BW} = \frac{\pi}{T_p\sqrt{1-\zeta^2}}\sqrt{(1-2\zeta^2)+\sqrt{4\zeta^4-4\zeta^2+2}}} \tag{10.56}$$

To relate the bandwidth to rise time, T_r, we use Figure 4.16, knowing the desired ζ and T_r. For example, assume $\zeta = 0.4$ and $T_r = 0.2$ second. Using Figure 4.16, the ordinate $T_r\omega_n = 1.463$, from which $\omega_n = 1.463/0.2 = 7.315$ rad/s. Using Eq. (10.54), $\omega_{BW} = 10.05$ rad/s. Normalized plots of Eqs. (10.55) and (10.56) and the relationship between bandwidth normalized by rise time and damping ratio are shown in Figure 10.41.

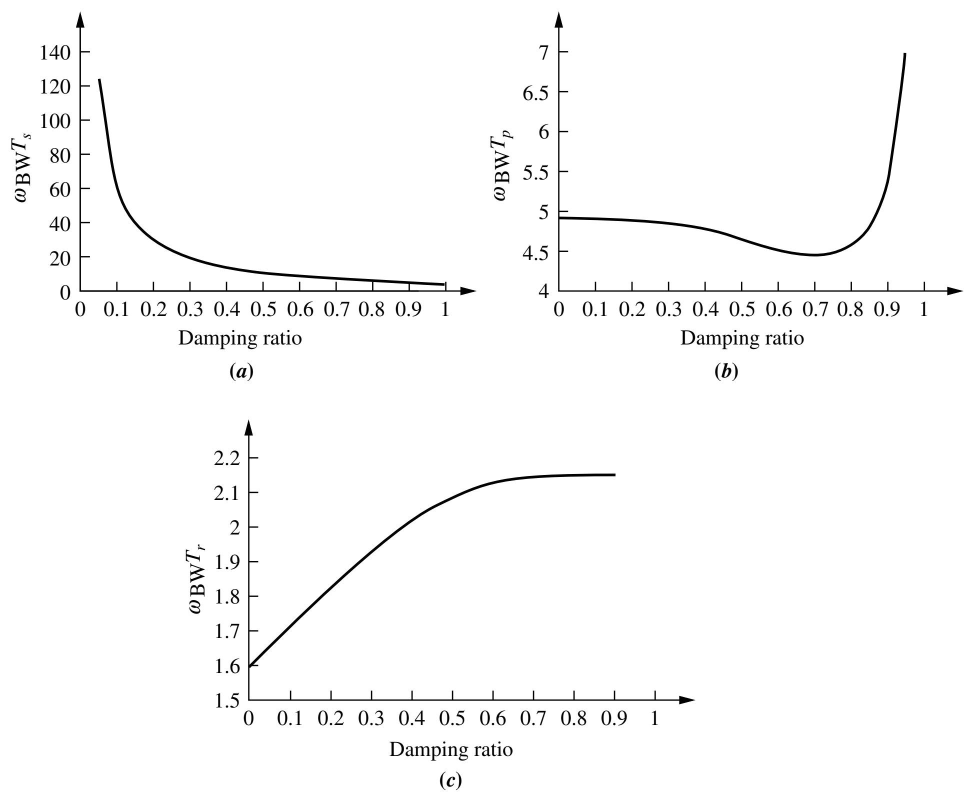

FIGURE 10.41 Normalized bandwidth vs. damping ratio for **a.** settling time; **b.** peak time; **c.** rise time

Skill-Assessment Exercise 10.7

PROBLEM: Find the closed-loop bandwidth required for 20% overshoot and 2-seconds settling time.

ANSWER: $\omega_{BW} = 5.79$ rad/s

The complete solution is at www.wiley.com/college/nise.

In this section, we related the closed-loop transient response to the closed-loop frequency response via bandwidth. We continue by relating the closed-loop frequency response to the open-loop frequency response and explaining the impetus.

10.9 Relation Between Closed- and Open-Loop Frequency Responses

At this point, we do not have an easy way of finding the closed-loop frequency response from which we could determine M_p and thus the transient response.[2] As we have seen, we are equipped to rapidly sketch the open-loop frequency response but not the closed-loop frequency response. However, if the open-loop response is related to the closed-loop response, we can combine the ease of sketching the open-loop response with the transient response information contained in the closed-loop response.

Constant *M* Circles and Constant *N* Circles

Consider a unity feedback system whose closed-loop transfer function is

$$T(s) = \frac{G(s)}{1+G(s)} \tag{10.57}$$

The frequency response of this closed-loop function is

$$T(j\omega) = \frac{G(j\omega)}{1+G(j\omega)} \tag{10.58}$$

Since $G(j\omega)$ is a complex number, let $G(j\omega) = P(\omega) + jQ(\omega)$ in Eq. (10.58), which yields

$$T(j\omega) = \frac{P(\omega)+jQ(\omega)}{[(P(\omega)+1)+jQ(\omega)]} \tag{10.59}$$

Therefore,

$$M^2 = |T^2(j\omega)| = \frac{P^2(\omega)+Q^2(\omega)}{[(P(\omega)+1)^2+Q^2(\omega)]} \tag{10.60}$$

Eq. (10.60) can be put into the form

$$\left(P+\frac{M^2}{M^2-1}\right)^2 + Q^2 = \frac{M^2}{(M^2-1)^2} \tag{10.61}$$

[2] At the end of this subsection, we will see how to use MATLAB to obtain closed-loop frequency responses.

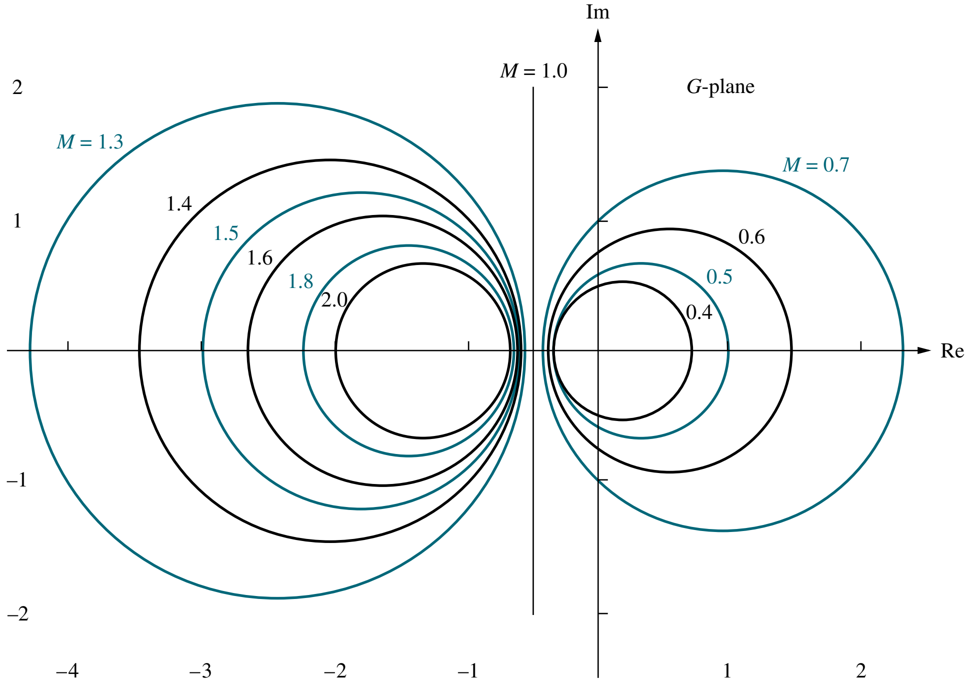

FIGURE 10.42 Constant M circles

which is the equation of a circle of radius $M/(M^2-1)$ centered at $[-M^2/(M^2-1),\ 0]$. These circles, shown plotted in Figure 10.42 for various values of M, are called *constant M circles* and are the locus of the closed-loop magnitude frequency response for unity feedback systems. Thus, if the polar frequency response of an open-loop function, $G(s)$, is plotted and superimposed on top of the constant M circles, the closed-loop magnitude frequency response is determined by each intersection of this polar plot with the constant M circles.

Before demonstrating the use of the constant M circles with an example, let us go through a similar development for the closed-loop phase plot, the constant N circles. From Eq. (10.59), the phase angle, ϕ, of the closed-loop response is

$$\begin{aligned}\phi &= \tan^{-1}\frac{Q(\omega)}{P(\omega)} - \tan^{-1}\frac{Q(\omega)}{P(\omega)+1} \\ &= \tan^{-1}\frac{\dfrac{Q(\omega)}{P(\omega)} - \dfrac{Q(\omega)}{P(\omega)+1}}{1+\dfrac{Q(\omega)}{P(\omega)}\left(\dfrac{Q(\omega)}{P(\omega)+1}\right)}\end{aligned} \tag{10.62}$$

after using $\tan(\alpha-\beta) = (\tan\alpha - \tan\beta)/(1+\tan\alpha\tan\beta)$. Dropping the functional notation,

$$\tan\phi = N = \frac{Q}{P^2+P+Q^2} \tag{10.63}$$

Equation (10.63) can be put into the form of a circle,

$$\left(P+\frac{1}{2}\right)^2 + \left(Q-\frac{1}{2N}\right)^2 = \frac{N^2+1}{4N^2} \tag{10.64}$$

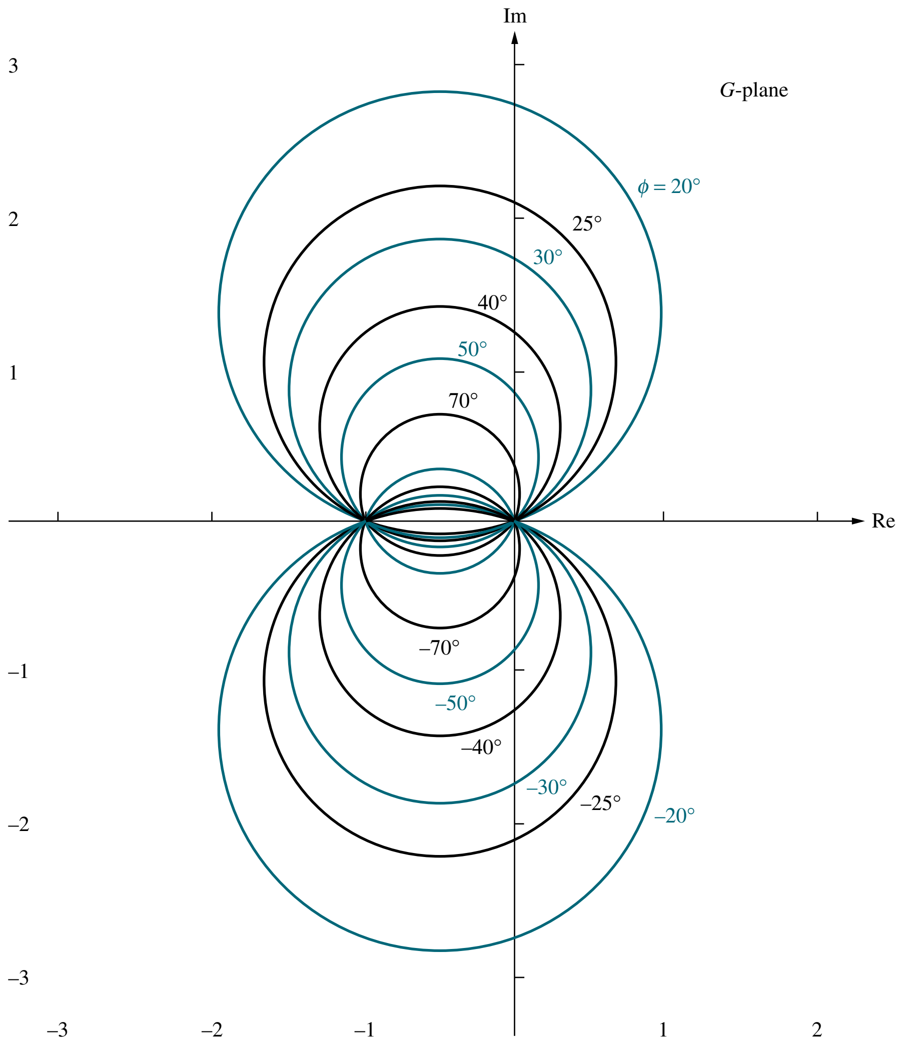

FIGURE 10.43 Constant N circles

which is plotted in Figure 10.43 for various values of N. The circles of this plot are called *constant N circles*. Superimposing a unity feedback, open-loop frequency response over the constant N circles yields the closed-loop phase response of the system. Let us now look at an example of the use of the constant M and N circles.

Example 10.11

Closed-Loop Frequency Response from Open-Loop Frequency Response

PROBLEM: Find the closed-loop frequency response of the unity feedback system shown in Figure 10.10, where $G(s) = 50/[s(s+3)(s+6)]$, using the constant M circles, N circles, and the open-loop polar frequency response curve.

SOLUTION: First evaluate the open-loop frequency function and make a polar frequency response plot superimposed over the constant M and N circles. The

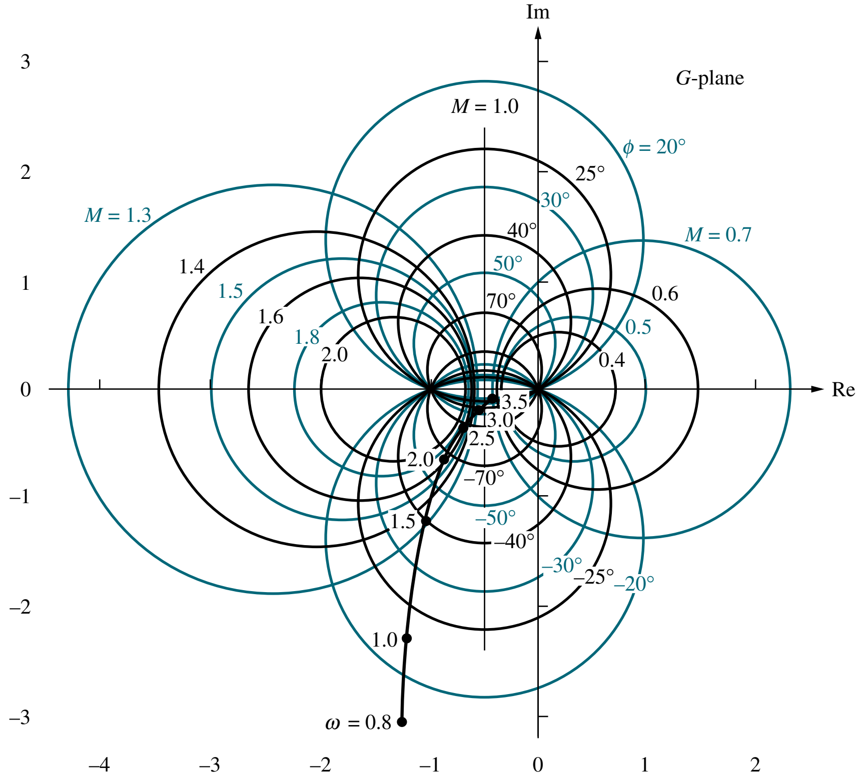

FIGURE 10.44 Nyquist diagram for Example 10.11 and constant M and N circles

open-loop frequency function is

$$G(j\omega) = \frac{50}{-9\omega^2 + j(18\omega - \omega^3)} \tag{10.65}$$

from which the magnitude, $|G(j\omega)|$, and phase, $\angle G(j\omega)$, can be found and plotted. The polar plot of the open-loop frequency response (Nyquist diagram) is shown superimposed over the M and N circles in Figure 10.44.

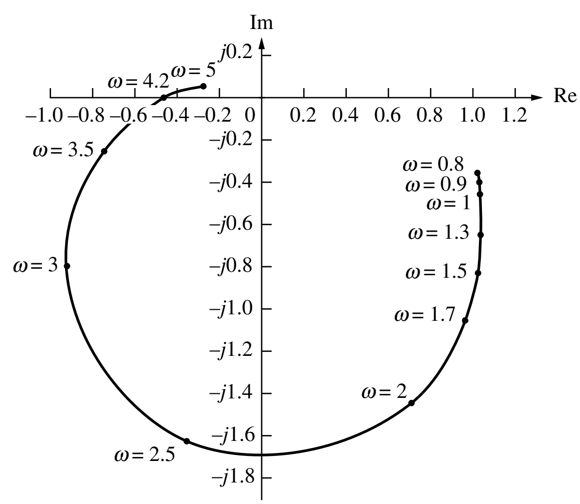

FIGURE 10.45 Closed-loop frequency response for Example 10.11

The closed-loop magnitude frequency response can now be obtained by finding the intersection of each point of the Nyquist plot with the M circles, while the closed-loop phase response can be obtained by finding the intersection of each point of the Nyquist plot with the N circles. The result is shown in Figure 10.45.[3]

`Students who are using MATLAB should now run ch10p5 in Appendix B. You will learn how to use MATLAB to find the closed-loop frequency response. This exercise solves Example 10.11 using MATLAB.`

MATLAB
ML

Nichols Charts

A disadvantage of using the M and N circles is that changes of gain in the open-loop transfer function, $G(s)$, cannot be handled easily. For example, in the Bode plot, a gain change is handled by moving the Bode magnitude curve up or down an amount equal to the gain change in dB. Since the M and N circles are not dB plots, changes in gain require each point of $G(j\omega)$ to be multiplied in length by the increase or decrease in gain.

Another presentation of the M and N circles, called a *Nichols chart*, displays the constant M circles in dB, so that changes in gain are as simple to handle as in the Bode plot. A Nichols chart is shown in Figure 10.46. The chart is a plot of open-loop magnitude in dB vs. open-loop phase angle in degrees. Every point on the M circles can be transferred to the Nichols chart. Each point on the constant M circles is represented by magnitude and angle (polar coordinates). Converting the magnitude to dB, we can transfer the point to the Nichols chart, using the polar coordinates with magnitude in dB plotted as the ordinate, and the phase angle plotted as the abscissa. Similarly, the N circles also can be transferred to the Nichols chart.

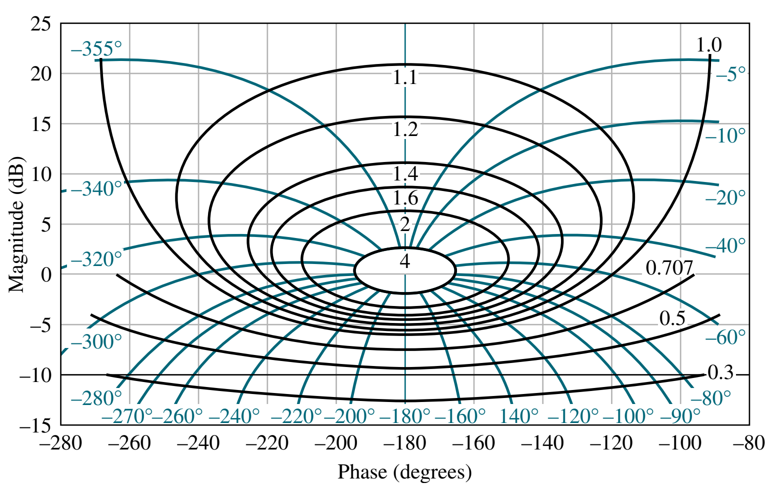

FIGURE 10.46 Nichols chart

[3] You are cautioned not to use the *closed-loop* polar plot for the Nyquist criterion. The closed-loop frequency response, however, can be used to determine the closed-loop transient response, as discussed in Section 10.8.