3

Mathematical Modeling of Mechanical Systems and Electrical Systems

3–1 INTRODUCTION

This chapter presents mathematical modeling of mechanical systems and electrical systems. In Chapter 2 we obtained mathematical models of a simple electrical circuit and a simple mechanical system. In this chapter we consider mathematical modeling of a variety of mechanical systems and electrical systems that may appear in control systems.

The fundamental law govering mechanical systems is Newton's second law. In Section 3–2 we apply this law to various mechanical systems and derive transfer-function models and state-space models.

The basic laws governing electrical circuits are Kirchhoff's laws. In Section 3–3 we obtain transfer-function models and state-space models of various electrical circuits and operational amplifier systems that may appear in many control systems.

3–2 MATHEMATICAL MODELING OF MECHANICAL SYSTEMS

This section first discusses simple spring systems and simple damper systems. Then we derive transfer-function models and state-space models of various mechanical systems.

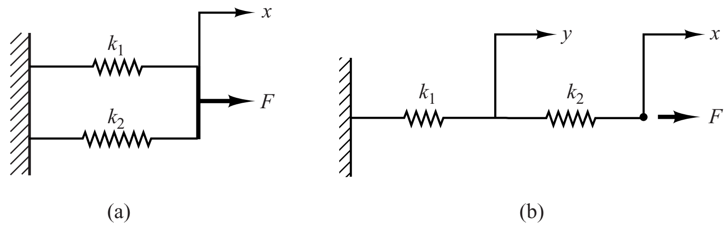

Figure 3–1
(a) System consisting of two springs in parallel;
(b) system consisting of two springs in series.

EXAMPLE 3–1 Let us obtain the equivalent spring constants for the systems shown in Figures 3–1(a) and (b), respectively.

For the springs in parallel [Figure 3–1(a)] the equivalent spring constant k_{eq} is obtained from

$$k_1 x + k_2 x = F = k_{eq} x$$

or

$$k_{eq} = k_1 + k_2$$

For the springs in series [Figure–3–1(b)], the force in each spring is the same. Thus

$$k_1 y = F, \qquad k_2(x - y) = F$$

Elimination of y from these two equations results in

$$k_2\left(x - \frac{F}{k_1}\right) = F$$

or

$$k_2 x = F + \frac{k_2}{k_1} F = \frac{k_1 + k_2}{k_1} F$$

The equivalent spring constant k_{eq} for this case is then found as

$$k_{eq} = \frac{F}{x} = \frac{k_1 k_2}{k_1 + k_2} = \frac{1}{\dfrac{1}{k_1} + \dfrac{1}{k_2}}$$

EXAMPLE 3–2 Let us obtain the equivalent viscous-friction coefficient b_{eq} for each of the damper systems shown in Figures 3–2(a) and (b). An oil-filled damper is often called a dashpot. A dashpot is a device that provides viscous friction, or damping. It consists of a piston and oil-filled cylinder. Any relative motion between the piston rod and the cylinder is resisted by the oil because the oil must flow around the piston (or through orifices provided in the piston) from one side of the piston to the other. The dashpot essentially absorbs energy. This absorbed energy is dissipated as heat, and the dashpot does not store any kinetic or potential energy.

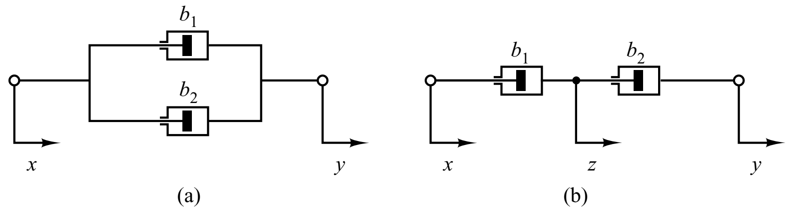

Figure 3–2 (a) Two dampers connected in parallel; (b) two dampers connected in series.

(a) The force f due to the dampers is

$$f = b_1(\dot{y} - \dot{x}) + b_2(\dot{y} - \dot{x}) = (b_1 + b_2)(\dot{y} - \dot{x})$$

In terms of the equivalent viscous-friction coefficient b_{eq}, force f is given by

$$f = b_{eq}(\dot{y} - \dot{x})$$

Hence

$$b_{eq} = b_1 + b_2$$

(b) The force f due to the dampers is

$$f = b_1(\dot{z} - \dot{x}) = b_2(\dot{y} - \dot{z}) \tag{3–1}$$

where z is the displacement of a point between damper b_1 and damper b_2. (Note that the same force is transmitted through the shaft.) From Equation (3–1), we have

$$(b_1 + b_2)\dot{z} = b_2\dot{y} + b_1\dot{x}$$

or

$$\dot{z} = \frac{1}{b_1 + b_2}(b_2\dot{y} + b_1\dot{x}) \tag{3–2}$$

In terms of the equivalent viscous-friction coefficient b_{eq}, force f is given by

$$f = b_{eq}(\dot{y} - \dot{x})$$

By substituting Equation (3–2) into Equation (3–1), we have

$$f = b_2(\dot{y} - \dot{z}) = b_2\left[\dot{y} - \frac{1}{b_1 + b_2}(b_2\dot{y} + b_1\dot{x})\right]$$

$$= \frac{b_1 b_2}{b_1 + b_2}(\dot{y} - \dot{x})$$

Thus,

$$f = b_{eq}(\dot{y} - \dot{x}) = \frac{b_1 b_2}{b_1 + b_2}(\dot{y} - \dot{x})$$

Hence,

$$b_{eq} = \frac{b_1 b_2}{b_1 + b_2} = \frac{1}{\dfrac{1}{b_1} + \dfrac{1}{b_2}}$$

EXAMPLE 3–3 Consider the spring-mass-dashpot system mounted on a massless cart as shown in Figure 3–3. Let us obtain mathematical models of this system by assuming that the cart is standing still for $t < 0$ and the spring-mass-dashpot system on the cart is also standing still for $t < 0$. In this system, $u(t)$ is the displacement of the cart and is the input to the system. At $t = 0$, the cart is moved at a constant speed, or $\dot{u} =$ constant. The displacement $y(t)$ of the mass is the output. (The displacement is relative to the ground.) In this system, m denotes the mass, b denotes the viscous-friction coefficient, and k denotes the spring constant. We assume that the friction force of the dashpot is proportional to $\dot{y} - \dot{u}$ and that the spring is a linear spring; that is, the spring force is proportional to $y - u$.

For translational systems, Newton's second law states that

$$ma = \sum F$$

where m is a mass, a is the acceleration of the mass, and $\sum F$ is the sum of the forces acting on the mass in the direction of the acceleration a. Applying Newton's second law to the present system and noting that the cart is massless, we obtain

$$m\frac{d^2y}{dt^2} = -b\left(\frac{dy}{dt} - \frac{du}{dt}\right) - k(y - u)$$

or

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = b\frac{du}{dt} + ku$$

This equation represents a mathematical model of the system considered. Taking the Laplace transform of this last equation, assuming zero initial condition, gives

$$\left(ms^2 + bs + k\right)Y(s) = (bs + k)U(s)$$

Taking the ratio of $Y(s)$ to $U(s)$, we find the transfer function of the system to be

$$\text{Transfer function} = G(s) = \frac{Y(s)}{U(s)} = \frac{bs + k}{ms^2 + bs + k}$$

Such a transfer-function representation of a mathematical model is used very frequently in control engineering.

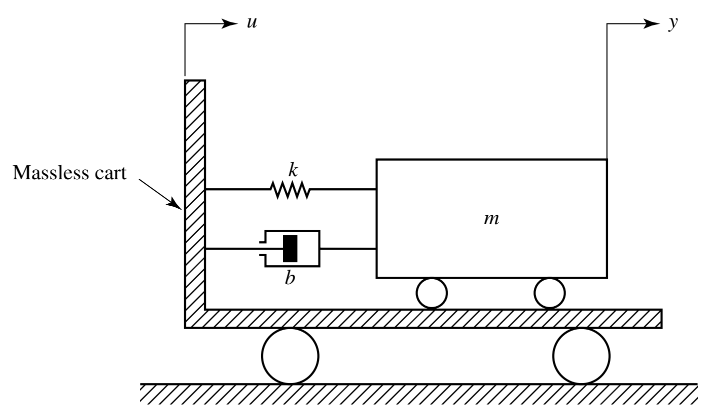

Figure 3–3 Spring-mass-dashpot system mounted on a cart.