Lecture 9: Time response – step response of 2nd order systems

- the unit step function was defined previously as:

$$\mathbb{H}(t)=\begin{cases}1, & for\ t>0\\ 0, & for\ t<0\end{cases} \tag{9.1}$$

- with Laplace transform $\mathbb{H}(s)=1/s$. The transfer function of a second order system is

$$G(s)=\frac{Y(s)}{U(s)}=\frac{\omega_n^{\ 2}}{s^2+2\xi\omega_n s+\omega_n^{\ 2}} \tag{9.2}$$

- thus, if the step input $u(t)=\mathbb{H}(t)$ was applied as input to this second-order system, the response function will be

$$Y(s)=U(s)G(s)=\frac{\omega_n^{\ 2}}{s\left(s^2+2\xi\omega_n s+\omega_n^{\ 2}\right)} \tag{9.3}$$

- using a partial fraction expansion, equation (9.3) becomes

$$Y(s)=\frac{1}{s}-\frac{s+2\xi\omega_n}{\left(s+\xi\omega_n\right)^2+\omega_d^{\ 2}} \tag{9.4}$$

- where $\omega_d=\omega_n\sqrt{1-\xi^2}$ is the ‘damped natural frequency’ or the ‘resonant frequency’ defined in the previous lecture.

- note that the first term on the right side of equation (9.4) comes from the *particular input* (a step in this case), whilst the second term contains information about the *system characteristics* only. Correspondingly, the inverse Laplace transforms of these terms yield the *particular solution* (the forced, or steady-state motion) and the *complementary solution* (the natural or transient motions).

9.1 Overdamped systems (ξ>1)

- for overdamped systems, we can write the response function (9.4) in the alternative form using a result from the previous lecture:

$$Y(s)=\frac{1}{s}-\frac{1}{\left(T_1-T_2\right)}\left[\frac{T_1}{\left(s+1/T_1\right)}-\frac{T_2}{\left(s+1/T_2\right)}\right] \tag{9.5}$$

- this response function has three poles: at $s=0$,

$$\begin{aligned} s_1&=-\xi\omega_n+\omega_n\sqrt{\xi^2-1}=-1/T_1\\ \&\ s_2&=-\xi\omega_n-\omega_n\sqrt{\xi^2-1}=-1/T_2\end{aligned} \tag{9.6}$$

- the inverse Laplace transform yields the unit step response:

$$y(t) = 1 - \frac{1}{T_1 - T_2}\left(e^{-t/T_1} - e^{-t/T_2}\right) \tag{9.7}$$

9.2 Critically damped systems (ξ=1)

- for critical damping, equation (9.4) becomes

$$Y(s) = \frac{1}{s} - \frac{s + 2\omega_n}{\left(s + \omega_n\right)^2} \tag{9.8}$$

- and the step response is

$$y(t) = 1 - \left(1 + \omega_n t\right)e^{-\omega_n t} \tag{9.9}$$

9.3 Underdamped systems (ξ<1)

- for underdamped systems, the inverse Laplace transform of equation (9.4) gives:

$$y(t) = 1 - \frac{\omega_n}{\omega_d} e^{-\xi\omega_n t} \sin\left(\omega_d t - \phi\right) \tag{9.10}$$

- where

$$\phi = \tan^{-1}\left(\frac{\omega_d}{-\xi\omega_n}\right) = \tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{-\xi}\right) \tag{9.11}$$

- an alternative expansion of equation (9.4) is

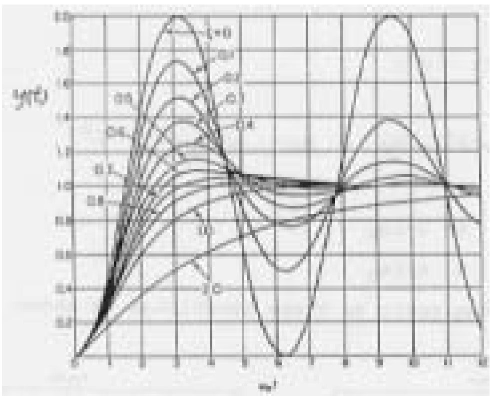
$$Y(s) = \frac{1}{s} - \frac{s + \xi\omega_n}{\left(s + \xi\omega_n\right)^2 + \omega_d^{\ 2}} - \frac{\xi\omega_n}{\left(s + \xi\omega_n\right)^2 + \omega_d^{\ 2}} \tag{9.12}$$

- which yields an alternative expression for the step response:

$$y(t) = 1 - e^{-\xi\omega_n t}\cos\left(\omega_d t\right) - \frac{\xi\omega_n}{\omega_d} e^{-\xi\omega_n t}\sin\left(\omega_d t\right) \tag{9.13}$$

- exercise: show that equations (9.10) and (9.13) are equivalent

- looking are the unit step responses defined by equation (9.4) for a range of values of damping ratio:



9.4 Time-domain performance specifications

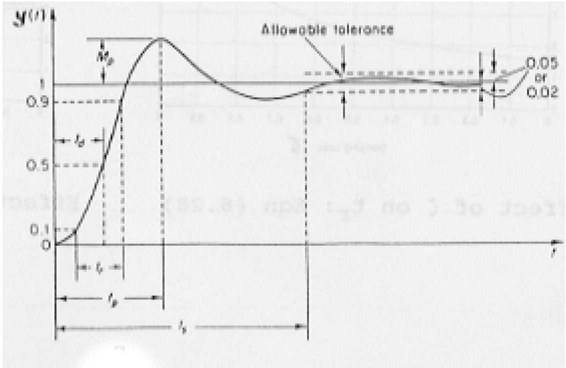
- it is common to express system performance requirements in terms of bounds on the zero-state response to a unit step input. Conventional definitions are

t_r = rise time

t_p = peak time

M_p = peak overshoot

t_s = settling time



- exercise: for the *underdamped* second-order system response defined by equations (9.10) or (9.13), show that the following relationships apply

1. <u>rise time</u> t_r (0-100%): the 0-100% rise time is convenient to calculate for an underdamped system. (Of course, it has no meaning for an overdamped system, for which the 10-90% rise time must be used.)

$$t_r = \frac{1}{\omega_d}\tan^{-1}\left(\frac{\omega_d}{-\xi\omega_n}\right) = \frac{\phi}{\omega_d} \simeq \frac{1.8}{\omega_n} \qquad (9.14)$$

2. <u>peak time</u> t_p:

$$t_p = \frac{\pi}{\omega_d} \qquad (9.15)$$

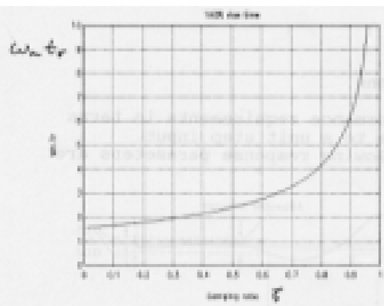
3. <u>peak overshoot</u> M_p (%):

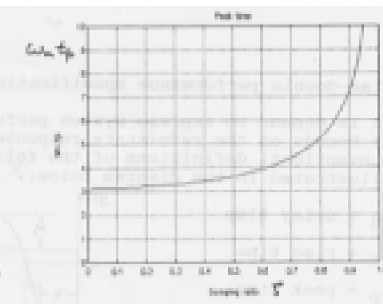
$$M_p(\%) = 100 e^{-\pi\xi\omega_n/\omega_d} \qquad (9.16)$$

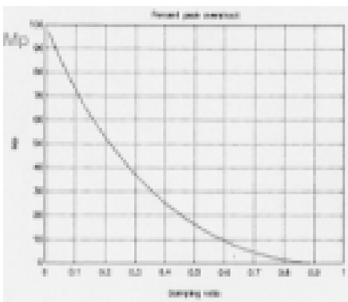
4. <u>settling time</u> t_s

$$t_s(5\%) \simeq 3T = 3/\xi\omega_n$$
$$\& \; t_s(2\%) \simeq 4T = 4/\xi\omega_n \qquad (9.17)$$

- the effect of damping ratio on the first three of these quantities can be seen in the following plots:



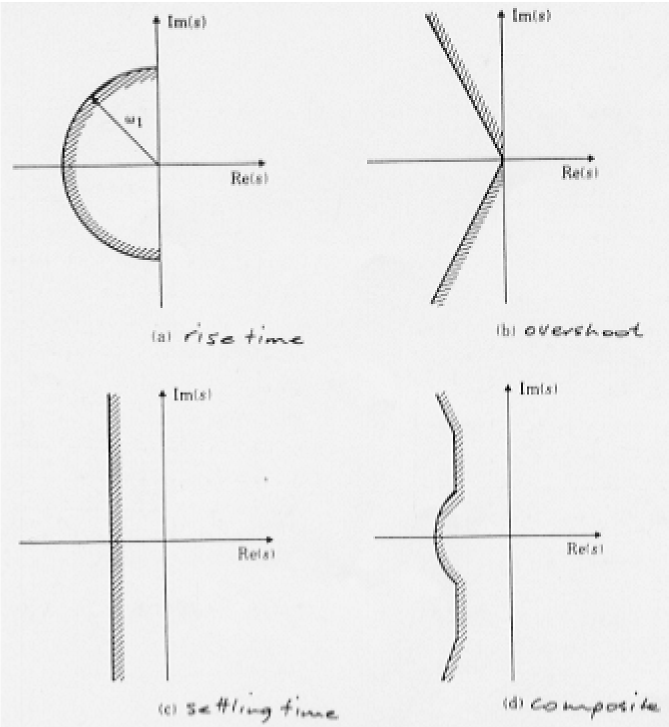




- the expression for the rise time (9.14) is often simplified to

$$t_r \simeq \frac{1.8}{\omega_n} \qquad (9.18)$$

- although this is only a very rough approximation. Using this expression for the rise time, specification of the response parameters can be mapped to allowable regions of the s-plane for the 'dominant second order poles':



- note that a system is said to have 'dominant second order poles' when a second order sub-system of the form (9.2) exists in the overall transfer function, and that this sub-system has significantly 'slower poles' than all others that exist in the overall transfer function.