Lecture 8: Time response – 2nd order systems

- in previous lectures we have seen a number of examples of second-order systems: RLC electrical circuits, the spring-mass-damper mechanical oscillator, etc. As we have seen, the transfer function for a simple second-order system may be written in the standard form

$$G(s) = \frac{Y(s)}{U(s)} = \frac{\omega_n^{\,2}}{s^2 + 2\xi\omega_n s + \omega_n^{\,2}} \tag{8.1}$$

- where $U(s)$ and $Y(s)$ are the Laplace transforms of the input and output, respectively, ξ is the damping ratio and ω_n is the undamped natural frequency.

8.1 Impulse response

- the zero-state response to a unit impulse reveals the natural characteristics of the system. That is, given a system at equilibrium $y(0) = \dot{y}(0) = 0$ at time $t = 0^-$, an impulsive input 'jars' the system to an initial velocity over an infinitesimal time. The subsequent motion from time $t = 0^+$ is unforced.

- exercise: show that the zero-input response to an initial 'velocity' $\dot{y}(0) = \omega_n^{\,2}$ is equivalent to the zero-state impulse response with $y(0) = \dot{y}(0) = 0$. This should help you appreciate the nature of impulse forcing.

- with $u(t) = \delta(t)$, it follows that $U(s) = 1$ and the response function $Y(s)$ is

$$Y(s) = G(s) = \frac{\omega_n^{\,2}}{s^2 + 2\xi\omega_n s + \omega_n^{\,2}} \tag{8.2}$$

- which expands to

$$Y(s) = G(s) = \frac{\omega_n}{2\sqrt{\xi^2 - 1}} \frac{1}{\left(s + \xi\omega_n - \omega_n\sqrt{\xi^2 - 1}\right)} - \frac{\omega_n}{2\sqrt{\xi^2 - 1}} \frac{1}{\left(s + \xi\omega_n + \omega_n\sqrt{\xi^2 - 1}\right)} \tag{8.3}$$

- the 'characteristic polynomial' $A(s)$ of the transfer function $G(s)$ is

$$A(s) = s^2 + 2\xi\omega_n + \omega_n^{\,2} \tag{8.4}$$

- and the roots of the 'characteristic equation' $A(s) = 0$ are

$$\begin{aligned} s_1 &= -\xi\omega_n + \omega_n\sqrt{\xi^2 - 1} \\ \& \; s_2 &= -\xi\omega_n - \omega_n\sqrt{\xi^2 - 1} \end{aligned} \tag{8.5}$$

- inverse Laplace transforming equation (8.3), the natural (ie. unforced) motion of the system is of the form

$$y(t) = Y(s) = G(s) = \frac{\omega_n}{2\sqrt{\xi^2 - 1}} e^{s_1 t} - \frac{\omega_n}{2\sqrt{\xi^2 - 1}} e^{s_2 t} \tag{8.6}$$

- note, therefore, that we need not always decompose the transfer function (8.2) into first order terms as in (8.3). Instead, we can gain information about the 'form' of the response simply by finding the roots of the characteristic equation $A(s) = 0$. The time response will then have the form

$$y(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t} \tag{8.7}$$

- which has the same form as equation (8.6), except we have not specified the values of the constants c_1, c_2

- in most cases, our interest is mainly in the form of the response (eg. is the system stable, unstable, oscillatory, etc). As equation (8.7) shows, this will depend only on whether the roots s_1, s_2 are real or complex, and is independent of the constants c_1, c_2. We call these roots the 'poles' of the transfer function (8.2)

8.2 Overdamped systems (ξ>1)

- if the damping ratio is larger than unity, the poles of the transfer function are real. Assuming for now that the poles are negative, they give rise to a decaying time response, and are therefore referred to as 'stable' poles:

$$\begin{aligned} s_1 &= -\xi\omega_n + \omega_n\sqrt{\xi^2 - 1} = -1/T_1 \\ \& \; s_2 &= -\xi\omega_n - \omega_n\sqrt{\xi^2 - 1} = -1/T_2 \end{aligned} \tag{8.8}$$

- the impulse response function is then

$$\begin{aligned} Y(s) = G(s) &= \frac{1}{T_1 T_2} \frac{1}{(s + 1/T_1)(s + 1/T_2)} \\ &= \frac{1}{(T_1 - T_2)} \left[\frac{1}{(s + 1/T_1)} - \frac{1}{(s + 1/T_2)} \right] \end{aligned} \tag{8.9}$$

- taking the inverse Laplace transform gives the impulse response:

$$y(t) = \frac{1}{T_1 - T_2} \left(e^{-t/T_1} - e^{-t/T_2} \right) \tag{8.10}$$

- example: the impulse response of an overdamped system with $\omega_n = \sqrt{2}$ and $\xi = 3/2\sqrt{2}$ has a characteristic polynomial

$$\begin{aligned} A(s) &= s^2 + 2\xi\omega_n s + \omega_n{}^2 \\ &= s^2 + 3s + 2 \end{aligned} \tag{8.11}$$

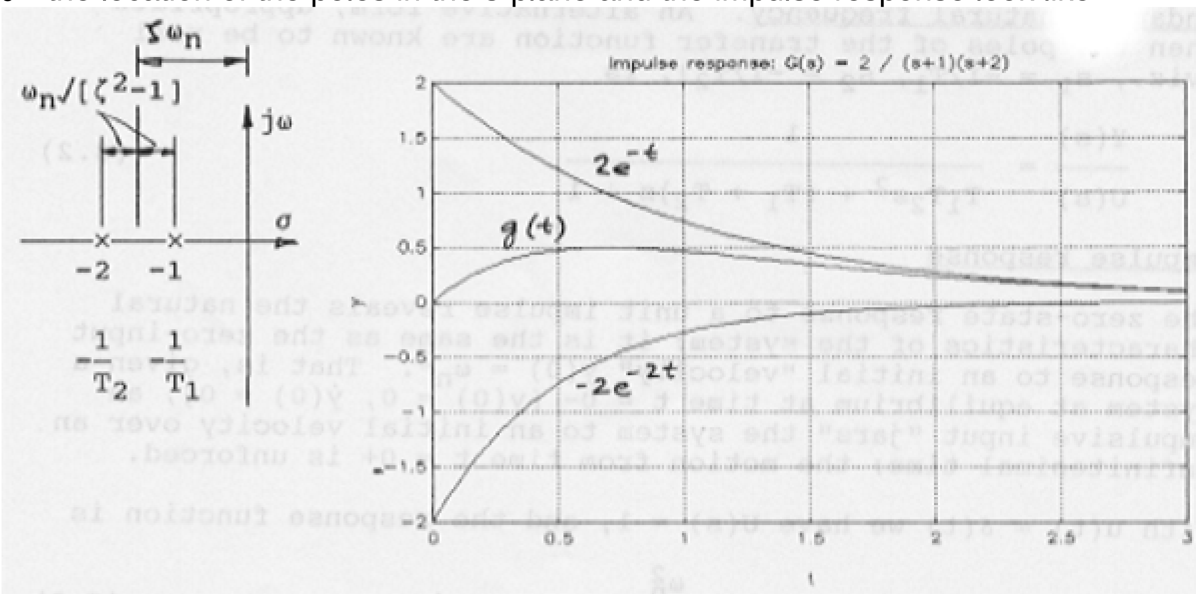
- and the transfer function is

$$G(s)=\frac{1}{s^2+3s+2}=\frac{1}{(s+1)(s+2)} \tag{8.12}$$

- from equation (8.9), we have in this case that $T_1=1, T_2=1/2$, and

$$y(t)=g(t)=2\left(e^{-t}-e^{-2t}\right) \tag{8.13}$$

- the location of the poles in the s-plane and the impulse response look like



8.3 Critically-damped systems (ξ=1)

- if the damping ratio is unity, the system poles are real and equal:

$$s_1=s_2=-\omega_n=-1/T \tag{8.14}$$

- the impulse response function is then

$$\begin{aligned}Y(s)=G(s)&=\frac{\omega_n^{\,2}}{(s+\omega_n)^2}\\&=\frac{1}{T^2}\frac{1}{(s+1/T)^2}\end{aligned} \tag{8.15}$$

- the inverse Laplace transform gives

$$\begin{aligned}y(t)=g(t)&=\omega_n^{\,2}te^{-t/T}\\&=\frac{1}{T^2}te^{-t/T}\end{aligned} \tag{8.16}$$

- the damping in this case is only "critical" in the sense that it defines the boundary between aperiodic and periodic natural motions.

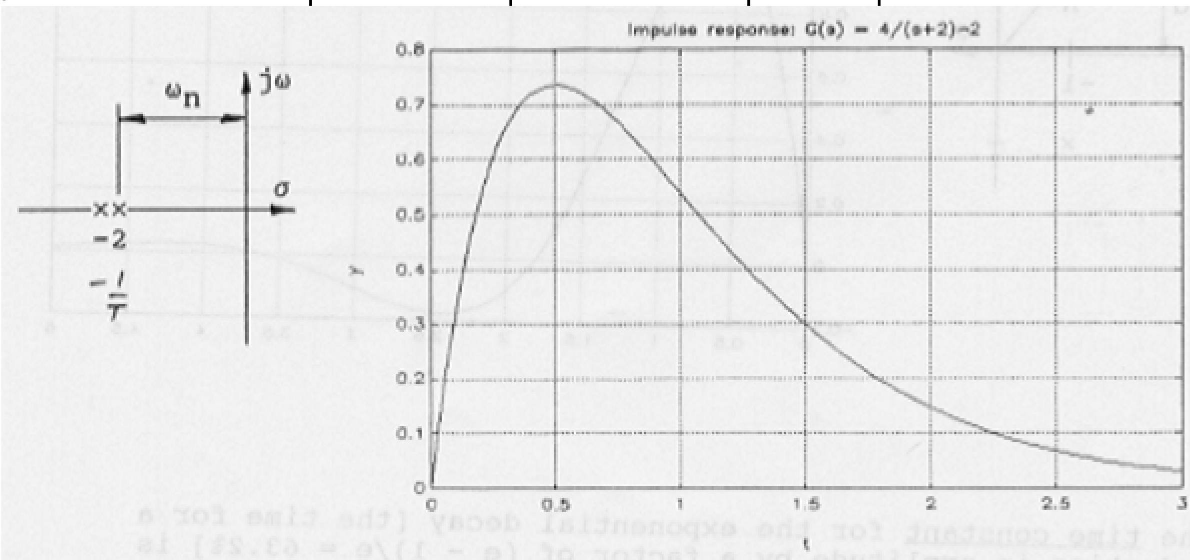
- example: the impulse response of a critically damped system with $\omega_n=2$. From equation (8.15), the impulse response function is

$$Y(s) = G(s) = \frac{4}{(s+2)^2} = \frac{4}{s^2 + 2s + 4} \tag{8.17}$$

- with the time response from(8.16):

$$y(t) = g(t) = 4te^{-2t} \tag{8.18}$$

- the location of the poles in the s-plane and the impulse response look like



8.4 Underdamped systems (ξ<1)

- if the damping ratio is less than unity, the system poles form a complex conjugate pair:

$$\begin{aligned} s_1 &= -\xi\omega_n + i\omega_d \\ \& \; s_2 &= -\xi\omega_n - i\omega_d \end{aligned} \tag{8.19}$$

- $\omega_d = \omega_n\sqrt{1-\xi^2}$ is the 'damped natural frequency'.

- the impulse response function is then

$$Y(s) = G(s) = \frac{\omega_n^{\,2}}{(s+\xi\omega_n)^2 + \omega_d^{\,2}} \tag{8.20}$$

- and impulse response from the inverse Laplace transform:

$$y(t) = \frac{\omega_n^{\,2}}{\omega_d} e^{-\xi\omega_n t} \sin(\omega_d t) \tag{8.21}$$

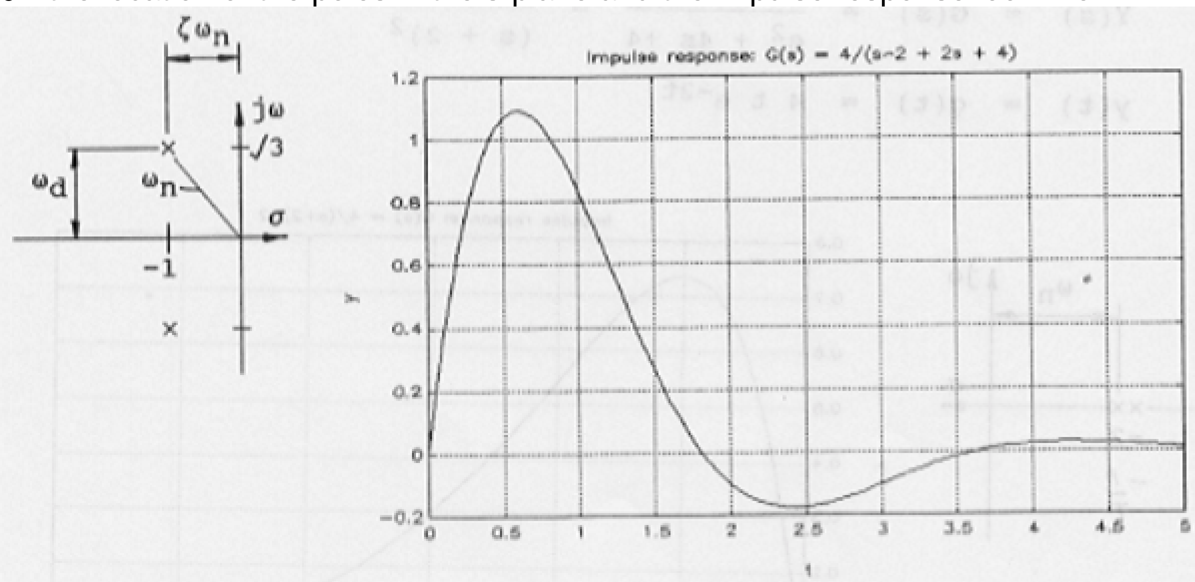
- example: the impulse response of an underdamped system with $\omega_n = 2, \xi = 1/2$. Using (8.20), the impulse response function is

$$Y(s) = G(s) = \frac{4}{s^2 + 2s + 4} = \frac{4}{(s+1)^2 + 3} \tag{8.22}$$

- with the time response from (8.21):

$$y(t) = \frac{4\sqrt{3}}{3} e^{-t} \sin\left(\sqrt{3}t\right) \tag{8.23}$$

- the location of the poles in the s-plane and the impulse response look like



- the time response in equation (8.23) can be seen to be the product of exponentially decaying and sinusoidal terms.

- the <u>time constant</u> for the exponential decay can be defined as the time for a reduction in amplitude by a factor of $1/e = 36.8\%$ is:

$$\frac{y_2(t)}{y_1(t)} = e^{-\xi\omega_n(t_2 - t_1)} = \frac{1}{e} \tag{8.24}$$

- thus,

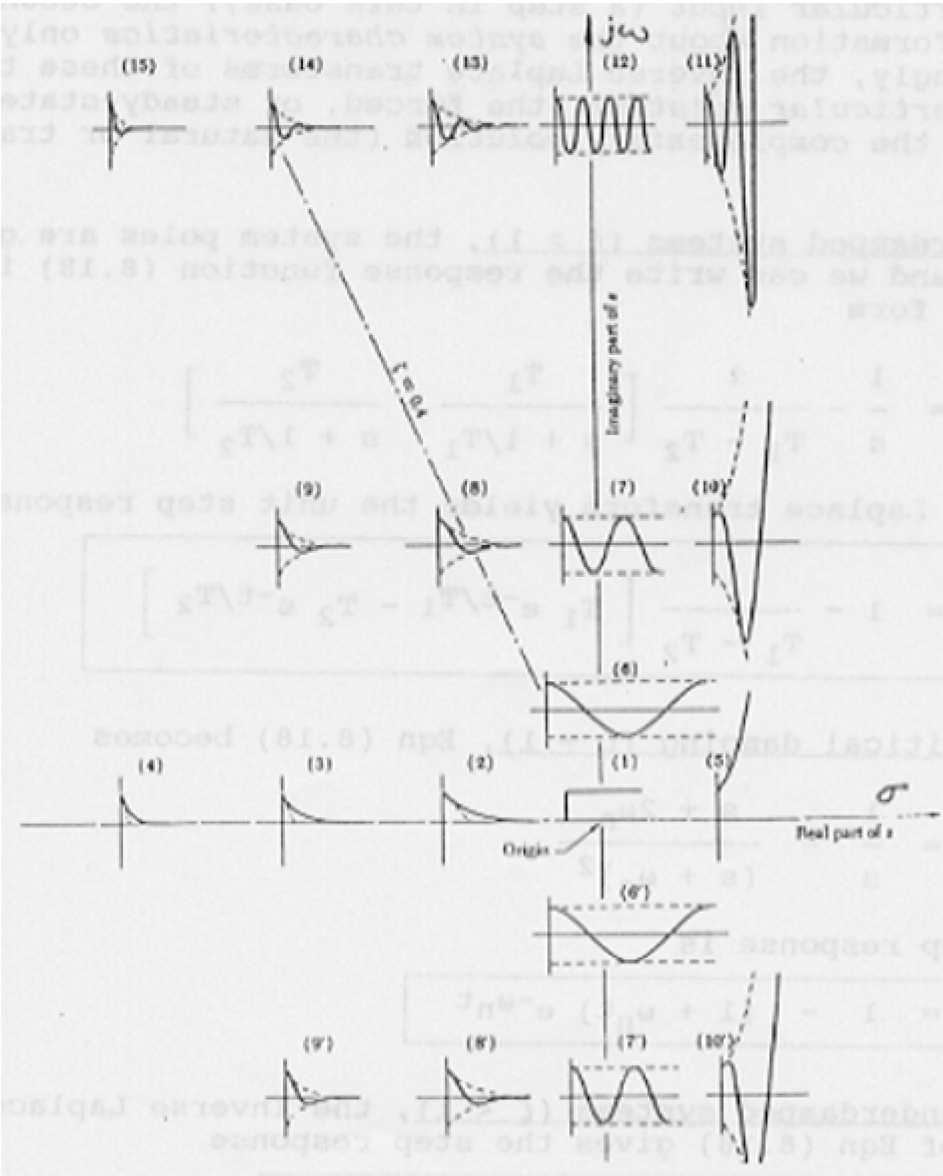
$$\begin{aligned} e^{\xi\omega_n(t_2 - t_1)} &= e^1 \\ \Rightarrow T = (t_2 - t_1) &= \frac{1}{\xi\omega_n} \end{aligned} \tag{8.25}$$

- the number of cycles of the damped oscillation required for the amplitude to decay by this factor depends on the damping ratio alone:

$$\begin{aligned} n &= \frac{\text{time constant}}{\text{period of osc.}} = \frac{T}{2\pi/\omega_d} = \frac{1}{\xi\omega_n}\frac{\omega_n\sqrt{1-\xi^2}}{2\pi} \\ &= \frac{\sqrt{1-\xi^2}}{2\pi\xi} \end{aligned} \tag{8.26}$$

8.5 The s-plane and the time response

- the examples of impulse responses given above indicated the location of the system poles in the s-plane. Establishing a mental correlation between the character of the natural motions and the s-plane pole locations is a useful aid to understanding the general response characteristics of a system without actually solving the equations, especially for higher-order systems (an nth-order system will have n poles). The following figure illustrates this correlation:



- note the following general features for poles with the form

$$s = \sigma + i\omega \tag{8.27}$$

1. if $\sigma > 0$ the system is 'unstable' since the time response *grows* exponentially
2. if $\sigma < 0$ the system is 'stable' since the time response *decays* exponentially
3. as $|\sigma| \to \infty$ the time response becomes faster because of the term $e^{\sigma t}$. Poles that are far from the imaginary axis are therefore often called 'fast', whilst those that are closer to the imaginary axis are called 'slow'.
4. as $|\omega| \to \infty$ the frequency of oscillation becomes faster.

- it is clear that we can determine a great deal about the form of the natural motions by considering the locations of poles in the s-plane!