Lecture 7: Time response – 1st order systems

- having developed a linear system model, we are now concerned with calculating the response of a system to initial conditions and to external inputs. We will begin by considering first-order systems. A first-order system with input u(t) and output y(t) may be represented by

  1. a differential equation
  $$T\dot{y}(t)+y(t)=Ku(t) \tag{7.1}$$

  2. the input u(t) and the unit impulse response
  $$g(t)=\frac{K}{T}e^{-t/T},\ t\geq 0 \tag{7.2}$$

  3. the Laplace-transformed input U(s) and a transfer function
  $$G(s)=\mathcal{L}\left[g(t)\right]=\frac{Y(s)}{U(s)}=\frac{K}{Ts+1} \tag{7.3}$$

- correspondingly, there are a number of ways of finding the response to some general input signal.

4.1 Classical solution

- the classical method for solving equation (7.1) is to say that the general solution is the sum of a *particular solution* (or *forced motion*) y_f and the *complementary solution* (or natural motion y_n:
$$y(t)=y_f(t)+y_n(t) \tag{7.4}$$

- where the forced motion satisfies (7.1) and the natural motion is a solution of the *homogeneous equation:*
$$T\dot{y}_n(t)+y_n(t)=0 \tag{7.5}$$

- the forced motion y_f is usually of the same *form* as the input: e.g. a sinusoidal input results in a sinusoidal forced response; a step input results in a constant forced response; a ramp input produces a linearly increasing forced response.

- consider the response to a typical test input: the Heaviside unit step $\mathbb{H}(t)$, where
$$u(t)=\mathbb{H}(t)=\begin{cases}1, & for\ t>0\\ 0, & for\ t<0\end{cases} \tag{7.6}$$

- equation (7.1) then becomes
$$T\dot{y}_f(t)+y_f(t)=K\mathbb{H}(t) \tag{7.7}$$

- the forced response is assumed to be of the same form as the input:
$$y_f=Y=\text{constant} \tag{7.8}$$

- substituting equation (7.8) into equation (7.7), we see that the differential equation is satisfied if Y = K, so that the forced response is

$$y_f = K \tag{7.9}$$

- the homogeneous equation is:

$$T\dot{y}_n(t) + y_n(t) = 0 \tag{7.10}$$

- we note that the function e^{st} has the property that its derivatives are constant multiples of the function itself, so a solution of this form has a chance of satisfying equation (7.10). We thus assume that the natural motion is of the form

$$y_n = Ce^{st} \tag{7.11}$$

- where C and s are (often complex) constants.

- substituting equation (7.11) into equation (7.10) gives

$$TsCe^{st} + Ce^{st} = Ce^{st}(Ts+1) = 0 \tag{7.12}$$

- apart from the trivial case C = 0 (for which there is no motion), equation (7.12) can be satisfied only if

$$Ts + 1 = 0 \tag{7.13}$$

- equation (7.13) is known as the characteristic equation of the system, because its *root* s = -1/T is the only value of s for which the assumed motion (7.11) can occur without external excitation. Thus the most general solution of the homogeneous equation (7.8) is the natural motion

$$y_n = Ce^{-t/T} \tag{7.14}$$

- and, from equation (7.4), the general solution of equation (7.7) for t > 0 is

$$y(t) = K + Ce^{-t/T} \tag{7.15}$$

- to determine C we need to introduce information about the initial condition(s). If we specify that $y(0) = y_0$ then equation (7.15) gives

$$C = y_0 - K \tag{7.16}$$

- and the complete solution is

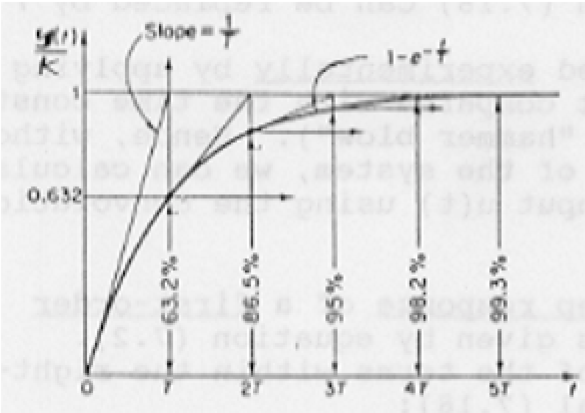
$$y(t) = \underbrace{K}_{\text{forced response}} + \underbrace{(y_0 - K)e^{-t/T}}_{\text{natural motion}} \tag{7.17}$$

- where we note that the forced response is the response to conditions *imposed* on the system from outside, whereas the natural motion is a *characteristic* of the system itself

- equation (7.17) may also be written as

$$y(t) = \underbrace{y_0 e^{-t/T}}_{\substack{zero-input \\ response}} + \underbrace{K\left(1-e^{-t/T}\right)}_{\substack{zero-state \\ response}} \tag{7.18}$$

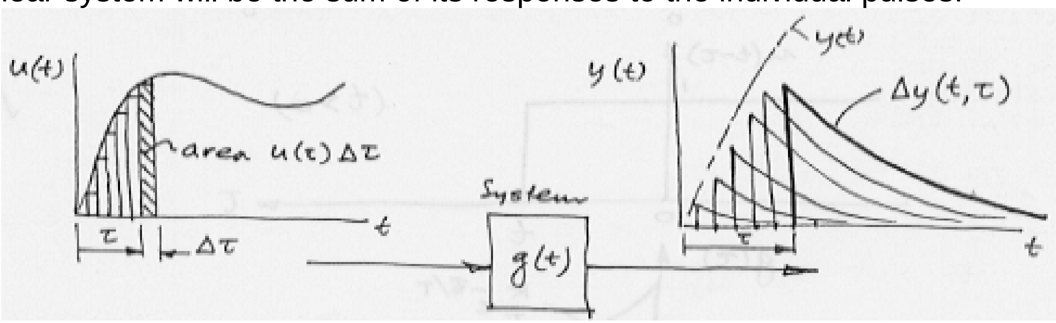
- ie. as the sum of the zero-input (no forced input) and the zero-state (zero initial conditions) responses. It is common to refer to the zero-state response to a unit step input as the unit step response:



$$\frac{y(t)}{K} = 1 - e^{-t/T}$$

4.2 Convolution solution

- an arbitrary continuous input $u(t)$ can be approximated by a staircase function, or a linear combination of shifted pulses, each of duration τ. The response of a linear system will be the sum of its responses to the individual pulses:



- if the width of the pulse occurring at time $t = \tau$ is very small compared to the time constants of the system, the pulse has the same effect as an impulse of strength $u(t)\Delta\tau$. The contribution to the system response from only this impulse is

$$\Delta y(t,\tau) = u(\tau)\Delta\tau g'(t-\tau) \tag{7.19}$$

- where $g'(t-\tau) = \delta(t-\tau)g(t-\tau)$ is the unit impulse response of the system and $\delta(t-\tau)$ is the impulse (or 'Dirac delta') function

$$\delta(t-\tau) = \begin{cases} \infty, & t = \tau \\ 0, & t \neq \tau \end{cases}$$
$$\& \int_{-\infty}^{+\infty} \delta(t-\tau)dt = 1 \tag{7.20}$$

- by superposition, the total zero-state response is then

$$y(t) \simeq \sum u(\tau)\Delta\tau g'(t-\tau) \tag{7.21}$$

- as $\Delta\tau \to 0$, and assuming the input is zero for t<0, we obtain the <u>convolution integral</u>:

$$y(t) = \int_0^t u(\tau)g'(t-\tau)\,d\tau = \int_0^t u(t-\tau)g'(t)\,d\tau \tag{7.22}$$

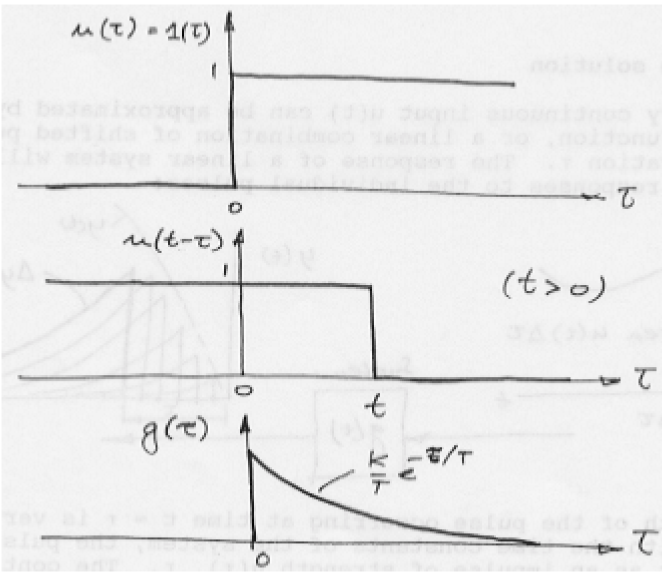
- the convolution integral is often written as

$$y(t) = u(t) * g(t) \tag{7.23}$$

- the alternative form of the integral on the right side of equation (7.22) can be found by a substitution of variables eg. $\beta = t - \tau$. Also, in the case that $g'(t-\tau) = \delta(t-\tau)g(t-\tau) = 0$ for $\tau > t$, the upper limit $\tau = t$ on the integrals in (7.22) can be replaced by $\tau = \infty$.

- note that $g'(t-\tau)$ can be determined <u>experimentally</u> by applying an pulse input which is very short compared with the time constants of the physical system (e.g. a "hammer blow"). Hence, without knowing the internal structure of the system, we can calculate its response to an arbitrary input $u(t)$ using the convolution integral (7.22).

- for our example of the unit step response of a first order system, $u(t) = \mathbb{H}(t)$ and $g'(t)$ is given by equation (7.2). Consider the variation with τ of the terms within the right-hand form of the convolution integral (7.22):



- we see that, for t > 0,

$$u(t-\tau)g(\tau)=\begin{cases} Ke^{-t/T}/T, & for\ 0<\tau<t \\ 0 & ,\quad otherwise \end{cases} \tag{7.24}$$

- hence

$$\begin{aligned} y(t) &= \int_0^t \frac{K}{T}e^{-\tau/T}d\tau \\ &= \left[-Ke^{-\tau/T}\right]_0^t \\ &= K\left(1-e^{-t/T}\right) \end{aligned} \tag{7.25}$$

- as obtained for the zero-state response previously in equation (7.18).

4.3 Laplace transform solution

- Laplace transforming equation (7.1), taking account of the initial condition $y(0)=y_0$

$$TsY(s)-Ty(0)+Y(s)=KU(s) \tag{7.26}$$

- thus,

$$\begin{aligned} Y(s) &= \frac{Ty_0}{Ts+1}+\frac{K}{Ts+1}U(s) \\ &= \frac{Ty_0}{Ts+1}+G(s)U(s) \end{aligned} \tag{7.27}$$

- where, as defined in an earlier lecture, $G(s)$ is the transfer function between $U(s)$ and $G(s)$

- for a unit step input $u(t)=\mathbb{H}(t)$ and $U(s)=1/s$. Hence equation (7.27) becomes

$$\begin{aligned} Y(s) &= \frac{Ty_0}{Ts+1}+\frac{K}{s(Ts+1)} \\ &= \frac{y_0}{s+1/T}+K\left(\frac{1}{s}-\frac{1}{s+1/T}\right) \end{aligned} \tag{7.28}$$

- inverse Laplace transforming equastion (7.28) using tables, we get

$$y(t)=y_0e^{-t/T}+K\left(1-e^{-t/T}\right) \tag{7.29}$$

- which is the same as that obtained previously in (7.18).

- note that the initial conditions are automatically accounted for and the zero-input solution comes from inverse-transforming the first term on the RHS of (7.28).

- if the initial conditions are zero, we can get the zero-state solution directly from the transfer function. For the present example, if we have our system model expressed in the form of the transfer function G(s) = Y(s)/U(s), the response function Y(s) can be obtained as

$$Y(s) = G(s)U(s) \quad (7.30)$$

- which for $U(s) = 1/s$ is

$$Y(s) = \frac{K}{(Ts+1)}\frac{1}{s} \quad (7.31)$$

- inverse-transforming equation(7.31), as in the right most terms of equations (7.28) and (7.29), yields the zero-state response $y(t) = K\left(1 - e^{-t/T}\right)$, as in equation (7.18).

- if the input is a unit impulse $u(t) = \delta(t)$, for which $U(s) = 1$, the response is by definition the unit impulse response $y(t) = g(t)$, which has a Laplace transform $Y(s) = \mathcal{L}\left[g(t)\right]$. From (7.30) we see that $Y(s) = G(s)$. Hence, *the transfer function is the Laplace transform of the unit impulse response*.