Lecture 7: Time response – 1st order systems

- having developed a linear system model, we are now concerned with calculating the response of a system to initial conditions and to external inputs. We will begin by considering first-order systems. A first-order system with input $u(t)$ and output $y(t)$ may be represented by

1. a differential equation
   \[ T\ddot{y}(t) + y(t) = Ku(t) \]  (7.1)

2. the input $u(t)$ and the unit impulse response
   \[ g(t) = \frac{K}{T} e^{-ut/T}, \quad t \geq 0 \]  (7.2)

3. the Laplace-transformed input $U(s)$ and a transfer function
   \[ G(s) = \mathcal{L}\left[ g(t) \right] = \frac{Y(s)}{U(s)} = \frac{K}{Ts + 1} \]  (7.3)

- correspondingly, there are a number of ways of finding the response to some general input signal.

4.1 Classical solution
- the classical method for solving equation (7.1) is to say that the general solution is the sum of a particular solution (or forced motion) $y_f$ and the complementary solution (or natural motion) $y_n$:
   \[ y(t) = y_f(t) + y_n(t) \]  (7.4)

- where the forced motion satisfies (7.1) and the natural motion is a solution of the homogeneous equation:
   \[ Ty_n(t) + y_n(t) = 0 \]  (7.5)

- the forced motion $y_f$ is usually of the same form as the input: e.g. a sinusoidal input results in a sinusoidal forced response; a step input results in a constant forced response; a ramp input produces a linearly increasing forced response.

- consider the response to a typical test input: the Heaviside unit step $\mathbb{H}(t)$, where
   \[ u(t) = \mathbb{H}(t) = \begin{cases} 1, & \text{for } t > 0 \\ 0, & \text{for } t < 0 \end{cases} \]  (7.6)

- equation (7.1) then becomes
   \[ T\ddot{y}_f(t) + y_f(t) = K\mathbb{H}(t) \]  (7.7)

- the forced response is assumed to be of the same form as the input:
   \[ y_f = Y = \text{constant} \]  (7.8)
substituting equation (7.8) into equation (7.7), we see that the differential equation is satisfied if \( Y = K \), so that the forced response is
\[
y_f = K
\] (7.9)

the homogeneous equation is:
\[
T\ddot{y}_n(t) + y_n(t) = 0
\] (7.10)

we note that the function \( e^{st} \) has the property that its derivatives are constant multiples of the function itself, so a solution of this form has a chance of satisfying equation (7.10). We thus assume that the natural motion is of the form
\[
y_n = Ce^{st}
\] (7.11)

where \( C \) and \( s \) are (often complex) constants.

substituting equation (7.11) into equation (7.10) gives
\[
TsCe^{st} + Ce^{st} = Ce^{st}(Ts + 1) = 0
\] (7.12)

apart from the trivial case \( C = 0 \) (for which there is no motion), equation (7.12) can be satisfied only if
\[
Ts + 1 = 0
\] (7.13)

equation (7.13) is known as the characteristic equation of the system, because its root \( s = -1/T \) is the only value of \( s \) for which the assumed motion (7.11) can occur without external excitation. Thus the most general solution of the homogeneous equation (7.8) is the natural motion
\[
y_n = Ce^{-st}
\] (7.14)

and, from equation (7.4), the general solution of equation (7.7) for \( t > 0 \) is
\[
y(t) = K + Ce^{-st}
\] (7.15)

to determine \( C \) we need to introduce information about the initial condition(s). If we specify that \( y(0) = y_0 \) then equation (7.15) gives
\[
C = y_0 - K
\] (7.16)

and the complete solution is
\[
y(t) = \frac{K}{\text{forced response}} + \left(y_0 - K\right)e^{-st}
\] (7.17)

where we note that the forced response is the response to conditions imposed on the system from outside, whereas the natural motion is a characteristic of the system itself

equation (7.17) may also be written as
\[ y(t) = y_0 e^{-t/T} + K \left( 1 - e^{-t/T} \right) \]  
(7.18)

- ie. as the sum of the zero-input (no forced input) and the zero-state (zero initial conditions) responses. It is common to refer to the zero-state response to a unit step input as the unit step response:

\[ \frac{y(t)}{K} = 1 - e^{-t/T} \]

### 4.2 Convolution solution

- an arbitrary continuous input \( u(t) \) can be approximated by a staircase function, or a linear combination of shifted pulses, each of duration \( \tau \). The response of a linear system will be the sum of its responses to the individual pulses:

\[ \text{if the width of the pulse occurring at time } t = \tau \text{ is very small compared to the time constants of the system, the pulse has the same effect as an impulse of strength } u(t) \Delta \tau. \text{ The contribution to the system response from only this impulse is} \]

\[ \Delta y(t, \tau) = u(\tau) \Delta \tau g'(t - \tau) \]  
(7.19)

- where \( g'(t - \tau) = \delta(t - \tau) g(t - \tau) \) is the unit impulse response of the system and \( \delta(t - \tau) \) is the impulse (or ‘Dirac delta’) function

\[
\delta(t - \tau) = \begin{cases} 
\infty, & t = \tau \\
0, & t \neq \tau 
\end{cases}
\]  
(7.20)

\[ \int_{-\infty}^{\infty} \delta(t - \tau) dt = 1 \]
by superposition, the total zero-state response is then

$$y(t) = \sum u(\tau) \Delta \tau g'(t-\tau)$$  \hspace{1cm} (7.21)

as $\Delta \tau \rightarrow 0$, and assuming the input is zero for $t<0$, we obtain the convolution integral:

$$y(t) = \int_0^t u(\tau) g'(t-\tau) d\tau = \int_0^t u(t-\tau) g'(t) d\tau$$  \hspace{1cm} (7.22)

the convolution integral is often written as

$$y(t) = u(t) * g(t)$$  \hspace{1cm} (7.23)

the alternative form of the integral on the right side of equation (7.22) can be found by a substitution of variables eg. $\beta = t-\tau$. Also, in the case that $g'(t-\tau) = \delta(t-\tau) g(t-\tau) = 0$ for $\tau > t$, the upper limit $\tau = t$ on the integrals in (7.22) can be replaced by $\tau = \infty$.

note that $g'(t-\tau)$ can be determined experimentally by applying an pulse input which is very short compared with the time constants of the physical system (e.g. a “hammer blow”). Hence, without knowing the internal structure of the system, we can calculate its response to an arbitrary input $u(t)$ using the convolution integral (7.22).

for our example of the unit step response of a first order system, $u(t) = H(t)$ and $g'(t)$ is given by equation (7.2). Consider the variation with $\tau$ of the terms within the right-hand form of the convolution integral (7.22):

we see that, for $t > 0$,
\[ u(t-\tau)g(\tau) = \begin{cases} Ke^{-\tau/T}/T, & \text{for } 0 < \tau < t \\ 0, & \text{otherwise} \end{cases} \] (7.24)

- hence

\[ y(t) = \int_0^T \frac{K}{T} e^{-\tau/T} d\tau \]
\[ = \left[ -Ke^{-\tau/T} \right]_0^T \]
\[ = K(1-e^{-T/T}) \] (7.25)

- as obtained for the zero-state response previously in equation (7.18).

4.3 Laplace transform solution

- Laplace transforming equation (7.1), taking account of the initial condition \( y(0) = y_0 \)

\[ TsY(s) - Ty(0) + Y(s) = KU(s) \] (7.26)

- thus,

\[ Y(s) = \frac{TY_0}{Ts + 1} + \frac{K}{Ts + 1}U(s) \]
\[ = \frac{TY_0}{Ts + 1} + G(s)U(s) \] (7.27)

- where, as defined in an earlier lecture, \( G(s) \) is the transfer function between \( U(s) \) and \( G(s) \)

- for a unit step input \( u(t) = \mathcal{H}(t) \) and \( U(s) = 1/s \). Hence equation (7.27) becomes

\[ Y(s) = \frac{TY_0}{Ts + 1} + \frac{K}{Ts + 1} \]
\[ = \frac{Y_0}{s + 1/T} + K \left( \frac{1}{s} - \frac{1}{s + 1/T} \right) \] (7.28)

- inverse Laplace transforming equation (7.28) using tables, we get

\[ y(t) = y_0 e^{-t/T} + K(1 - e^{-t/T}) \] (7.29)

- which is the same as that obtained previously in (7.18).

- note that the initial conditions are automatically accounted for and the zero-input solution comes from inverse-transforming the first term on the RHS of (7.28).

- if the initial conditions are zero, we can get the zero-state solution directly from the transfer function. For the present example, if we have our system model
expressed in the form of the transfer function $G(s) = Y(s)/U(s)$, the response function $Y(s)$ can be obtained as

$$Y(s) = G(s)U(s)$$  \hfill (7.30)

- which for $U(s) = 1/s$ is

$$Y(s) = \frac{K}{(Ts + 1)} \frac{1}{s}$$  \hfill (7.31)

- inverse-transforming equation (7.31), as in the right most terms of equations (7.28) and (7.29), yields the zero-state response $y(t) = K(1 - e^{-t/T})$, as in equation (7.18).

- if the input is a unit impulse $u(t) = \delta(t)$, for which $U(s) = 1$, the response is by definition the unit impulse response $y(t) = g(t)$, which has a Laplace transform $Y(s) = \mathcal{L}[g(t)]$. From (7.30) we see that $Y(s) = G(s)$. Hence, the transfer function is the Laplace transform of the unit impulse response.