Lecture 6: Modelling – mechanical systems

- physical insight into the behaviour of mechanical systems is in general best obtained if the differential equations of motion are obtained by applying:

1. Newtonian mechanics
   - define coordinates required to specify configuration of system
   - draw free body diagrams of system components showing all acting forces, including component interaction forces
   - write Newton's laws for each free body
   - write constitutive, compatibility and constraint equations
   - use compatibility equations to eliminate unwanted internal interaction force variables.

2. Lagrangian mechanics
   - define generalised coordinates
   - define system boundary
   - write expressions for kinetic energy, potential energy and dissipation function
   - evaluate Lagrange's equation for each generalised coordinate.

6.1 Mechanical networks
- mechanical networks can be constructed using ‘mechanical impedances’. The mechanical impedance is defined as the ratio of the applied force to the relative velocity over the element.

- for a mass element, Newton gives us \( F(t) = mx \), thus
  \[
  Z(s) = \frac{F(s)}{V(s)} = \frac{ms^2X(s)}{sX(s)} = ms
  \]  
  (6.1)

- for a spring element, we have \( F(t) = kx \), thus
  \[
  Z(s) = \frac{F(s)}{V(s)} = \frac{kX(s)}{sX(s)} = \frac{k}{s}
  \]  
  (6.2)

- for a damper element, we have \( F(t) = c\dot{x} \), thus
  \[
  Z(s) = \frac{F(s)}{V(s)} = \frac{csX(s)}{sX(s)} = c
  \]  
  (6.3)

- given these basic definitions, we can now build up mechanical networks using simple rules that are analogous to those used in analysis electrical networks.

- when analysing mechanical networks the following combination rules apply:
1. **Series connection**

- using the relative velocities \( V_1 \) and \( V_2 \) across each element
  \[ F = Z_1 V_1 = Z_2 V_2 \]  
  \((6.4)\)
- hence \( V_2 / V_1 = Z_2 / Z_1 \) and
  \[ V = V_1 + V_2 \]
  \[ = V_1 (1 + V_2 / V_1) \]
  \[ = V_1 (1 + Z_2 / Z_1) \]  
  \((6.5)\)
- thus, for mechanical impedances in series
  \[ Z = \frac{F}{V} = \frac{Z_1 V_1}{1} \frac{1}{V_1 (1 + Z_2 / Z_1)} \]
  \[ = \frac{Z_1 Z_2}{(1 + Z_2 / Z_1)} \]  
  \((6.6)\)

2. **Parallel connection**

- clearly
  \[ F = F_1 + F_2 \]  
  \((6.7)\)
- where \( F_1 = Z_1 V \) and \( F_2 = Z_2 V \), giving
  \[ F = (Z_1 + Z_2) V \]  
  \((6.8)\)
- thus, for mechanical impedances in parallel
  \[ Z = \frac{F}{V} = Z_1 + Z_2 \]  
  \((6.9)\)

### 6.2 Example

- the equivalent mechanical network is
to find the transfer impedance $F/V_2$, consider a simplified network diagram:

- from (6.6) we have
  \[
  \frac{F_2}{V_1} = \frac{Z_2Z_3}{Z_2 + Z_3}
  \]  
  \[(6.10)\]

- while, by definition
  \[
  \frac{F_2}{V_2} = Z_3
  \]  
  \[(6.11)\]

- so that
  \[
  \frac{V_1}{V_2} = \frac{Z_2 + Z_3}{Z_2}
  \]  
  \[(6.12)\]

- whilst
  \[
  \frac{F}{V_1} = Z_1 + \frac{Z_2Z_3}{Z_2 + Z_3}
  \]  
  \[(6.13)\]

- equations (6.12) and (6.13) then yield
  \[
  \frac{F}{V_2} = Z_3 + \frac{Z_1(Z_2 + Z_3)}{Z_2}
  \]  
  \[
  = m_2s + \frac{k_2}{s} + \left(\frac{m_1s + k_1}{s}\right)\left(\frac{K + C + m_2s + k_2}{s}\right)
  \]  
  \[
  = m_2s + \frac{k_2}{s} + \frac{K}{s+C}
  \]  
  \[(6.14)\]
the transfer function of interest may be the \( \text{receptance } X_2(s)/F(s) \), where \( X_2 = \frac{V_2}{s} \). This follows from (6.14):

\[
\frac{X_2}{F} = \frac{V_2}{sF} = \frac{(Cs + K)}{(m_1s^2 + k_1)(m_2s^2 + Cs + k_2 + K) + (m_2s^2 + k_2)(Cs + K)}
\]

(6.15)