Lecture 5: Modelling – electrical systems

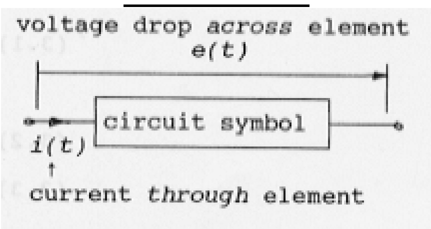
5.1 Relationship with other subjects

- control theory is applied to improve the performance of engineering systems involving many diverse physical elements, including mechanical, electrical, thermal and fluid systems. Knowledge about the technologies employed in these elements, and the basic theory for modelling them, comes from your other subjects; in this subject we are concerned with first developing *system* models, in a form which is amenable to system analysis and control design.

5.2 Electrical systems

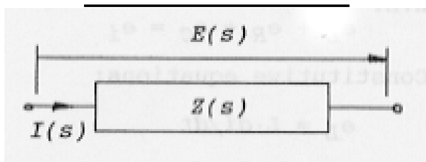
- as well as writing differential 'equations of motion' using the Kirchhoff current and voltage laws and the constitutive equations shown in Table IV, it is quite common with electrical systems to work directly in the (Laplace) frequency domain using 'operational impedances'.

Time domain



- constitutive equation:
  $$a(D)e(t) = b(D)i(t)$$
- where a(D), b(D) are differential operators; $D \equiv d/dt$

Frequency domain



- Laplace transform constitutive equation:
  $$a(s)E(s) = b(s)I(s)$$

Inductor

$$e = L{\cdot}di/dt$$

- i.e.: e = Ldi

$$E(s) = Ls{\cdot}I(s)$$

- where $Z_L = Ls$

Resistor

$$e = R{\cdot}i$$

$$E(s) = R{\cdot}I(s)$$

- where $Z_R = R$

Capacitor

$$C{\cdot}de/dt = i$$

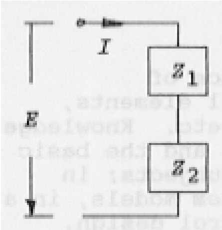
- i.e.: CDe = I

$$CsE(s) = I(s)$$

- where $Z_C = 1/Cs$

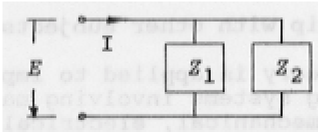
- with operational impedances, we may use the familiar rules for serial and parallel combinations:

<u>Series</u>



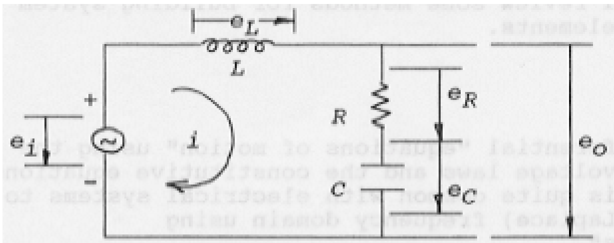
$$Z_0 = e/i = Z_1 + Z_2$$

<u>Parallel</u>



$$Z_0 = e/i = \frac{Z_1 Z_2}{\left(Z_1 + Z_2\right)}$$

5.3 Example: Finding a circuit transfer function



(i) <u>Using differential equations</u>

- using Kirchoff's voltage law ('KVL' – the equilibrium equation):

$$e_L + e_R + e_C = e_i \tag{5.1}$$

- the constitutive equations for each element are

$$\begin{aligned} e_L &= L\frac{di}{dt} \\ e_R &= Ri \\ C\frac{de_C}{dt} = i \quad &\Rightarrow e_C = \frac{1}{C}\int i dt + e_C(0) \end{aligned} \tag{5.2}$$

- the output potential (the constraint equation):

$$e_R + e_C = e_O \tag{5.3}$$

- substituting (5.2) into (5.1) gives

$$\begin{aligned} L\frac{di}{dt} + Ri + \frac{1}{C}\int i dt + e_C(0) &= e_i \\ Ri + \frac{1}{C}\int i dt + e_C(0) &= e_o \end{aligned} \tag{5.4}$$

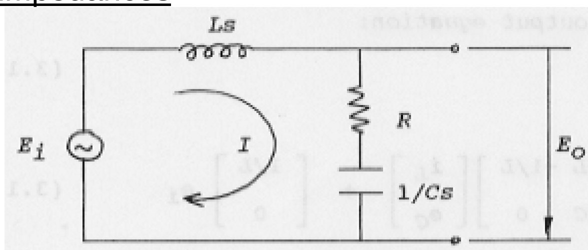
- Laplace transforming (5.4), with zero i.c.s gives

$$\left(Ls + R + \frac{1}{Cs}\right) I(s) = E_i(s)$$
$$\left(R + \frac{1}{Cs}\right) I(s) = E_o(s) \tag{5.5}$$

- hence

$$\frac{E_o(s)}{E_i(s)} = \frac{\left(R + \frac{1}{Cs}\right)}{\left(Ls + R + \frac{1}{Cs}\right)} = \frac{(RCs + 1)}{(LCs^2 + RCs + 1)} \tag{5.6}$$

(ii) Using operational impedances



- considering the circuit as a voltage divider:

$$\frac{E_0}{E_i} = \frac{Z_R + Z_C}{Z_L + Z_R + Z_C} \tag{5.7}$$

- thus:

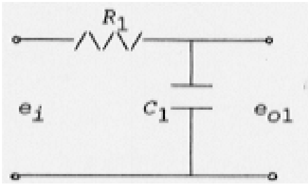
$$\frac{E_0}{E_i} = \frac{R + \frac{1}{Cs}}{Ls + R + \frac{1}{Cs}} \tag{5.8}$$

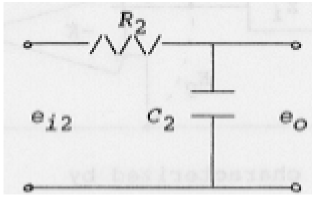
- which is the same as (5.6) above.

5.4 Cascaded transfer functions

- the transfer function of two cascaded (series) elements can be found as the product of the individual transfer functions, *provided that the second element does not 'load' the first* (the effect of loading is discussed further next semester*).* In electrical terms, loading does not occur if the input impedance of the second element is very high (and ideally infinite) compared to the output impedance of the first element.

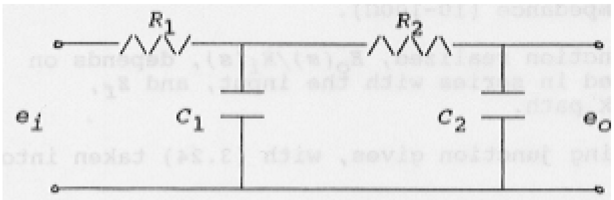
- for example, consider two filter circuits:





$$G_1(s) = \frac{E_{o1}}{E_i} = \frac{1}{R_1 C_1 s + 1} \qquad G_2(s) = \frac{E_o}{E_{i2}} = \frac{1}{R_2 C_2 s + 1}$$

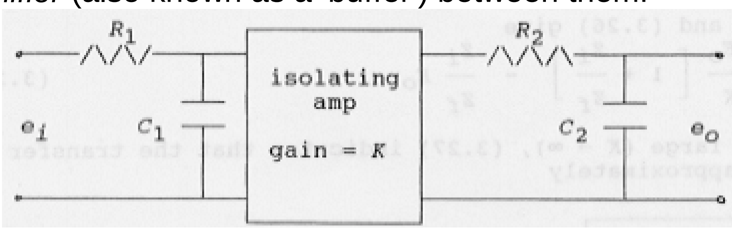
- the transfer functions given are for an open-circuit output in both cases, which can also be considered to be a 'virtual' infinite impedance. If the two circuits are cascaded, the second "loads" the first:



- <u>exercise</u>: show that the transfer function of the cascaded filters is

$$G_O(s) = \frac{E_O(s)}{E_i(s)} = \frac{1}{(R_1 C_1 s + 1)(R_2 C_2 s + 1) + R_1 C_2 s} \neq G_1(s) G_2(s) \qquad (5.9)$$

- if we wanted to cascade the two filters without loading the first, we could insert an *isolating amplifier* (also known as a 'buffer') between them:



- the transfer function is then:

$$G_O(s) = \frac{E_O(s)}{E_i(s)} = \frac{K}{(R_1 C_1 s + 1)(R_2 C_2 s + 1)} = K G_1(s) G_2(s) \qquad (5.10)$$

- the isolating amplifier in the above example could be implemented with an *operational amplifier* (which will be studied in greater detail in second semester and in other subjects).