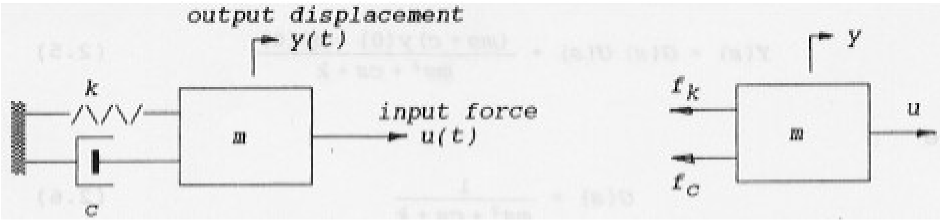
Lecture 3: Modelling - Representation

To illustrate the various ways in which a mathematical model of a system may be presented, consider a simple ‘mechanical oscillator’:

Physical model | Free body diagram



3.1 Derivation of mathematical model

‘equilibrium’ equation:

$$u - f_k - f_c = m\ddot{y} \tag{3.1}$$

‘constitutive’ equations for the spring and the damper:

The equation set (3.1), (3.2) then constitutes one form of the mathematical model.

$$\begin{aligned} f_k &= ky \\ f_c &= c\dot{y} \end{aligned} \tag{3.2}$$

3.2 Differential equation representation

Substituting (3.2) into (3.1) gives

$$m\ddot{y}(t) + c\dot{y}(t) + ky(t) = u(t) \tag{3.3}$$

The mechanical oscillator is therefore an example of a ‘second-order’ system i.e. one which is modelled by a second-order differential equation, as in (3.3).

3.3 Transfer function representation

Laplace transforming (3.3) gives:

$$m\left[s^2Y(s) - sy(0) - \dot{y}(0)\right] + c\left[sY(s) - y(0)\right] + kY(s) = U(s) \tag{3.4}$$

Where the (unilateral) Laplace transform of *y (t)* is

$$Y(s) = \mathcal{L}[y(t)] = \int_0^\infty y(t)e^{-st}dt \tag{3.5}$$

Then,

$$\left[ms^2 + cs + k\right]Y(s) = U(s) + \left[ms + c\right]y(0) + m\dot{y}(0) \tag{3.6}$$

Thus, the *differential* relationship between *u (t)* and *y (t)* has been transformed into an *algebraic* relationship between *U(s)* and *Y(s)*, which can be solved for *Y(s)*:

$$Y(s) = G(s)U(s) + \frac{\left[(ms + c)y(0) + m\dot{y}(0)\right]}{\left[ms^2 + cs + k\right]} \tag{3.7}$$

Where,

$$G(s) = \frac{1}{ms^2 + cs + k} \tag{3.8}$$

Is the transfer function from *u* to *y*. *Note that the transfer function is the ratio of the response function Y(s) to the input function U(s) <u>when all initial conditions are assumed to be zero</u>:*

$$G(s) = \left(\frac{Y(s)}{U(s)}\right)_{i.c=0} \tag{3.9}$$

When the form of the input *u(t) is* specified, its Laplace transform *U(s)* may be substituted in (3.7), and the output *y(t)* obtained by inverse Laplace transforming this equation.

3.4 Block diagram representation

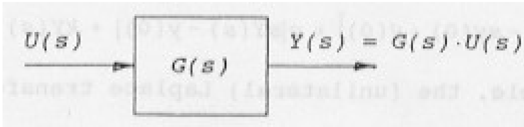
A block diagram is a graphical representation of the system model, which can reveal the 'structure' of the model more readily than a set of equations. A block diagram consists of:

1. <u>Blocks</u>, which represent the input-output behaviour of *independent* components of the system,

2. <u>Directed lines</u>, which represent the (uni-directional) signal flows between blocks, and

3. <u>Summing junctions</u>, which represent the addition and/or subtraction of signals.

A block is usually labelled with the *transfer function* of the component it represents, and the lines entering and leaving a block are labelled with symbols representing the *Laplace transforms* of the input and output signals:



It is conventional to have the *system input* signal enter from the left of the diagram, and the *system output* signal exit to the right. Then, the flow of signals in the 'forward paths' of the system are generally from left to right, while the 'feedback paths' generally flow from right to left.

There is no unique block diagrammatic representation of a given system. Diagrams may be rearranged, and blocks combined into larger components, according to simple rules of 'block diagram algebra'. Ultimately, any block diagram may be reduced to a single block, which represents the overall transfer function of the system.

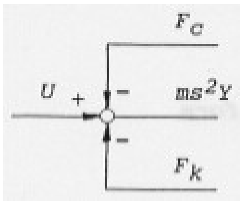
A block diagram may be constructed from the Laplace-transformed equation set. In our example, Laplace transforming (3.1) and (3.2) (assuming zero initial conditions) yields

$$U - F_k - F_c = ms^2 Y \tag{3.10}$$

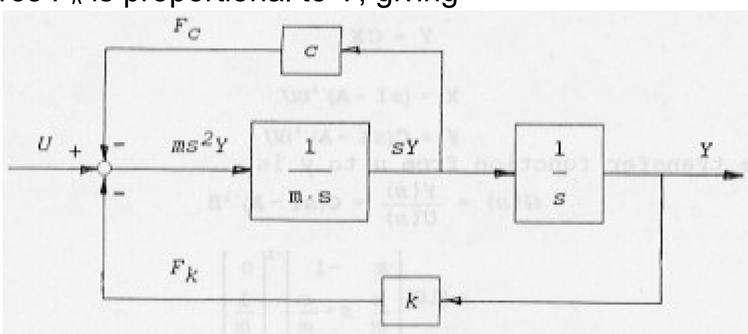
And

$$\begin{aligned} F_k &= kY \\ F_c &= csY \end{aligned} \tag{3.11}$$

eq. (3.10) can be represented thus:



From (2.9), the (Laplace-transformed) dashpot force *Fc* is proportional to the (transformed) velocity *sY*, which may be obtained by integrating the output from the summing junction and scaling the signal by 1/*m*. The transfer function for an integrator is 1/*s*. Similarly, the displacement *Y* is the integral of the velocity, and the spring force F_k is proportional to *Y*, giving



This detailed block diagram reveals the feedback structure in the dynamics of the mechanical oscillator.

3.5 Significance of the Laplace transform and the transfer function

The significance of the Laplace transform representation of signals, and the transfer function representation of linear system dynamics, lie in the fact that

1. The transfer function *G(s)* shows how signals of the form e^{st} are *transferred* through the system, and

2. The Laplace transform *U(s)* shows the "contribution" to the signal *u(t)* made by the elementary signal e^{st}, for any value of *s*.

Note that if the input to a linear system is of the form e^{st}, the output will be of the same form, simply scaled in magnitude and possibly time-shifted (because derivatives of the function *est* are constant multiples of the function itself).

Thus, if the input is

$$u(t) = Ue^{st} \tag{3.12}$$

The output is

$$y(t) = G(s)Ue^{st} \tag{3.13}$$

Where both *s* and *U* may be complex.

the way that linear systems transfer signals of the form e^{st} is of particular interest because this function is the fundamental ‘building block’ for creating any physically-realizable signal in a dynamical system.

Thus, if an input signal can be constructed as a linear combination of components of the form e^{st}, superposition allows us to construct the output signal from the responses to the each of the elementary components.

For example, if

$$u(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t} + c_3 e^{s_3 t} \tag{3.14}$$

Then

$$y(t) = c_1 G(s_1) e^{s_1 t} + c_2 G(s_2) e^{s_2 t} + c_3 G(s_3) e^{s_3 t} \tag{3.15}$$

In the following sections, we shall see how more complex signals may be Constructed-using e^{st} as a building block.