Lecture 2: Modelling

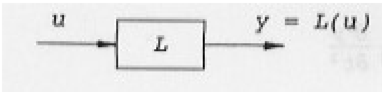
2.1 Complexity

For control *design,* we generally seek the simplest model, which represents the important aspects of the system's behaviour. We neglect "small effects".

Note that feedback control *can* make the closed-loop dynamics relatively insensitive to modelling errors. One thrust of control theory is to design control laws that are *robust* to these model uncertainties and *adaptive* to changes in the control system.

For *simulation,* we generally want a more accurate model in order to check on the effect of details on the actual system performance, and to verify that the simplifying assumptions made for control design purposes were justified.

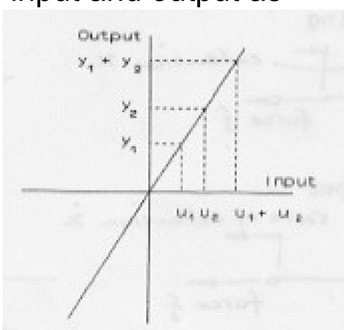
2.2 Linearity



A system is generally said to be *linear* if *superposition* holds:

$$L\,(c_1 u_1 + c_2 u_2) = c_1 L\,(u_1) + c_2 L\,(u_2) \tag{2.1}$$

With the relationship between input and output as



A linear system:

1. Can be described by *linear equations* (often-differential equations);

   An equation is linear if the dependent variable/s (and it/their derivatives) appear to the first power only, and there are no products of dependent variables.

   Example linear o.d.e.:

$$3\frac{d^2y}{dt^2} + 14\frac{dx}{dt} = 5t^3 \qquad (2.2)$$

Example nonlinear o.d.e:

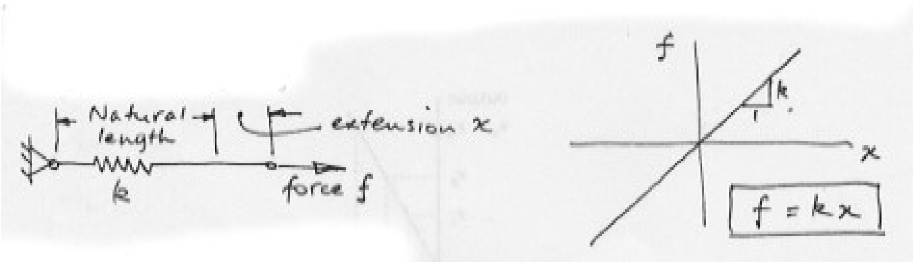
$$m\ddot{y} + ky + \underbrace{\alpha y^3}_{\substack{nonlinear \\ term}} = f(t) \qquad (2.3)$$

Example linear p.d.e. (wave equation):

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} \qquad (2.4)$$

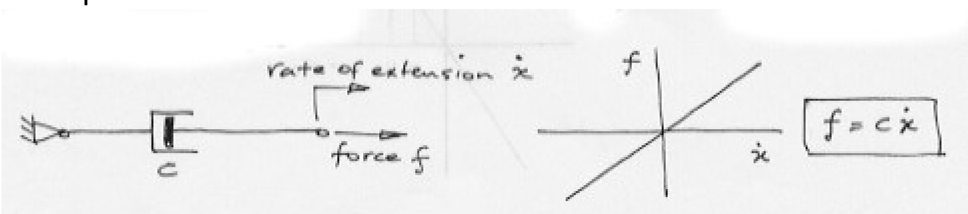
2. Will be composed of an assembly of *linear elements* (or subsystems):

e.g., ideal spring:



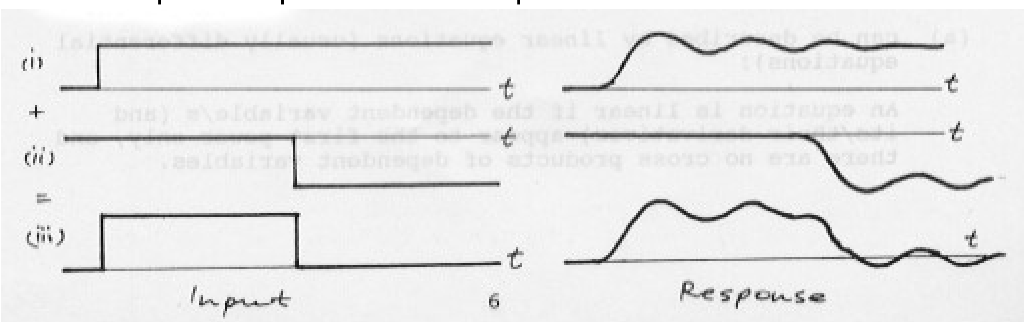
$$F = kx \qquad (2.5)$$

ideal damper



$$F = cx \qquad (2.6)$$

3. Will have a response to complicated inputs which is the *sum* (superposition) of the responses to simpler components of the input

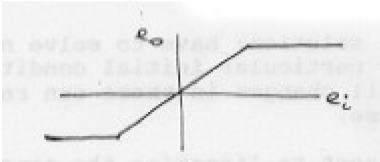


2.3 Common nonlinear elements

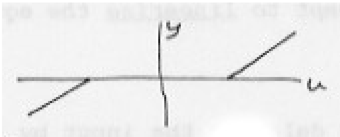
nonlinear spring

nonlinear damper

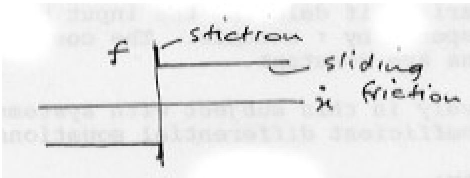
Saturation



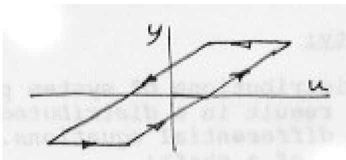
Deadband



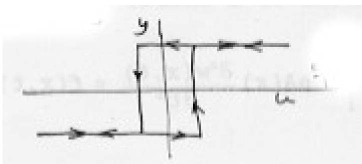
Coulomb friction



Hysteresis

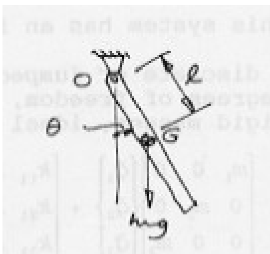


on-off element with hysteresis



geometric nonlinearity

e.g. the simple pendulum



With equation of motion

$$I_0\ddot{\theta} + mg\ell\sin\theta = 0 \tag{2.7}$$

Can linearize for small θ: sin θ≈ θ, giving

$$I_0\ddot{\theta} + mg\ell\,\theta = 0 \qquad (2.8)$$

2.4 Advantages of a linear model

- Can solve equations analytically.
- Once solved, solution is valid for all magnitudes of motion.
- Superposition allows construction of response to complex inputs from responses to simpler components of input (e.g. Fourier series).
- System "response characteristics" are consistent: experience with linear systems leads to transfer of knowledge to new situations; often we do not need to solve explicitly for the time history, because we can predict the general form of response from the system characteristics.

2.5 Typical features of nonlinear systems:

Generally no analytical solution; have to solve numerically.

Solution valid only for particular initial conditions and magnitude of input: small changes in these can result in large changes in response.

Hence, we shall generally attempt to linearize the equations of mathematical models whenever it is reasonable to do so.

2.6 Time variability

A system is time-invariant if delaying the input by seconds merely delays the response by seconds. The coefficients in the differential equations are constant.

We will deal exclusively in this subject with systems described by linear constant coefficient differential equations. These systems are often referred to as LTI (linear, time invariant) systems.

2.7 Spatial continuity

Continuous spatial distributions of properties (such as mass or capacitance) result in a *distributed-parameter* model, described by *partial differential equations.*

2.8 Temporal continuity

In *continuous-time* systems, signals (inputs, outputs, internal variables) are defined for all time in the interval [0, ∞). They are described by *differential* equations.

In discrete-*time* (or sampled-data) systems, signals are defined only at discrete instants of time (the sample times) -- digital control systems, for example. They are described by *difference* equations.

In this subject we deal with continuous-time systems only.

2.9 Determinism

If the system parameters and inputs are known, we can develop a *deterministic* model. If the parameters are uncertain, or the inputs vary randomly, we need to employ *stochastic* control theory, employing a probabilistic representation of signals and parameters.

This subject is restricted to deterministic models.

2.10 Modelling procedure

1. Specify the system to be modelled. Identify the system boundary, across which the *system* interacts with its *environment*.

2. Specify a physical model, which is an abstraction that exhibits the essential aspects of the real system's behaviour. This step involves engineering judgement.

3. Derive a mathematical model to represent the physical model. This may involve system identification - use of experimental data to determine the structure and parameters of the mathematical model.

4. Analyze the dynamic behaviour of the system and modify the model if necessary to better represent the actual system.

Derivation of the mathematical model requires specification of the physical variables with which to describe the configuration or state of the system (e.g., displacements, forces, currents), and will generally involve consideration of the following set of equations:

1. Equilibrium equations, such as Newton's and Kirchhoff's laws.

2. Constraint or compatibility equations, such as geometric constraints or interconnection relationships.
3. Constitutive equations describing the properties of elements in the system, such as a force-deflection relationship for a spring.

for analysis and control design purposes, the equations of motion derived for the physical model will generally be presented in one of the following forms:

1. A *transfer function* relating the Laplace Transform of an "output" variable to the Laplace Transform of an "input" variable.

2. A single (high-order) differential equation for the output variable.

3. A *block diagram*, which displays the structure and relationships of the equation set.

4. *State-space equations*: a set of first-order differential equations in which the independent variables are "state variables".