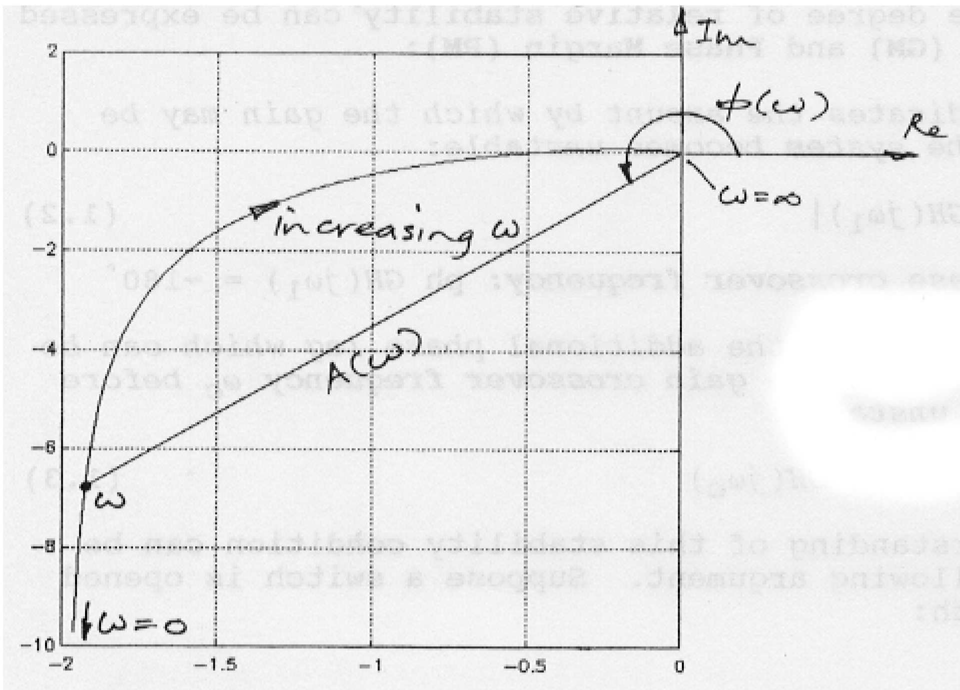
Lecture 19: Nyquist plots

- the frequency response of the OLTF may also be displayed as a polar plot in an Argand diagram.

- for example, consider again the OLTF from the previous lecture:

$$GH(s) = \frac{K}{s(s+1)^2} \qquad (19.1)$$

- where $A(\omega) = |GH(\omega)|$ and $\phi(\omega) = ph[GH(i\omega)]$. A polar plot of $A(\omega)$ vs $\phi(\omega)$ for $K = 1$ gives:



19.1 Nyquist stability criterion

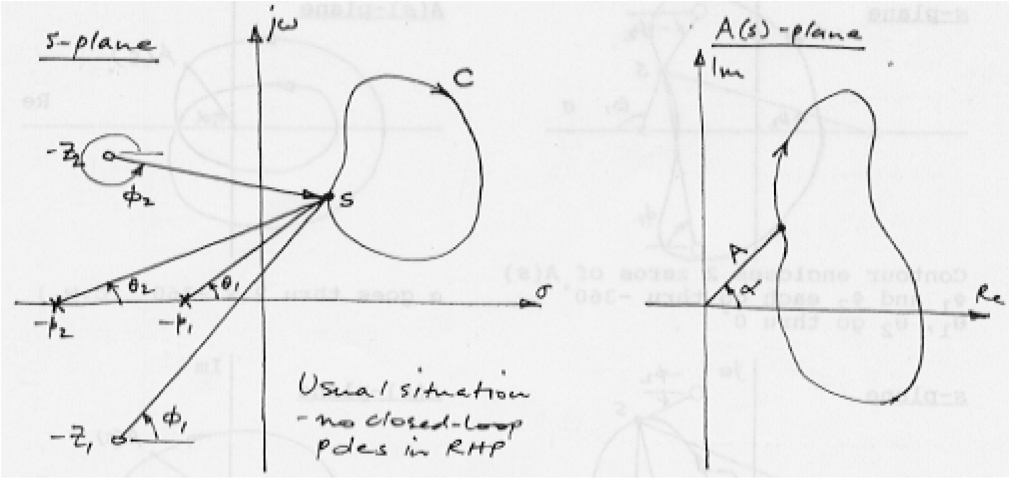
- this stability test employs the OLTF displayed in polar form, and is based on Cauchy's argument principle from complex variable theory.

- consider the characteristic equation:

$$\begin{aligned} A(s) = 1 + GH(s) &= 1 + \frac{K(s+z_1)(s+z_2)\dots(s+z_m)}{(s+p_1)(s+p_2)\dots(s+p_n)} \\ &= \frac{K(s+z_1)(s+z_2)\dots(s+z_n)}{(s+p_1)(s+p_2)\dots(s+p_n)} \end{aligned} \qquad (19.2)$$

- note that the numerator of $A(s)$ is of the same order as the denominator, and that the **poles** of $A(s)$ are the same as the open-loop poles. Stability is determined by the closed-loop poles, which are the **zeros** of $A(s)$.

- suppose now that the poles and zeros of some $A(s)$ are as follows in the *s*-plane, and that $A(s)$ is evaluated for values of *s* on an arbitrary closed contour in the *s*-plane. The contour may enclose poles or zeros, but not pass through them. This contour will map into the $A(s)$ plane as a closed curve:



- we can write the characteristic equation in the following form

$$A(s)=\frac{K|s+z_1||s+z_2|\ldots|s+z_n|}{|s+p_1||s+p_2|\ldots|s+p_n|}e^{i(\phi_1+\phi_2+\cdots+\phi_n-\theta_1-\theta_2\cdots-\theta_n)} \tag{19.3}$$

- thus, $A(s)$ can be written in terms of a magnitude and phase $A(s)=|A(s)|e^{i\alpha}$ with the phase given by

$$\begin{aligned}\alpha = ph\left[A(s)\right] &= ph(s+z_1)+ph(s+z_2)\ldots+ph(s+z_n)\\ &\quad -ph(s+p_1)-ph(s+p_2)\ldots-ph(s+p_n)\\ &=\phi_1+\phi_2+\cdots+\phi_n-\theta_1-\theta_2-\theta_n\end{aligned} \tag{19.4}$$

- in terms of the example, this becomes:

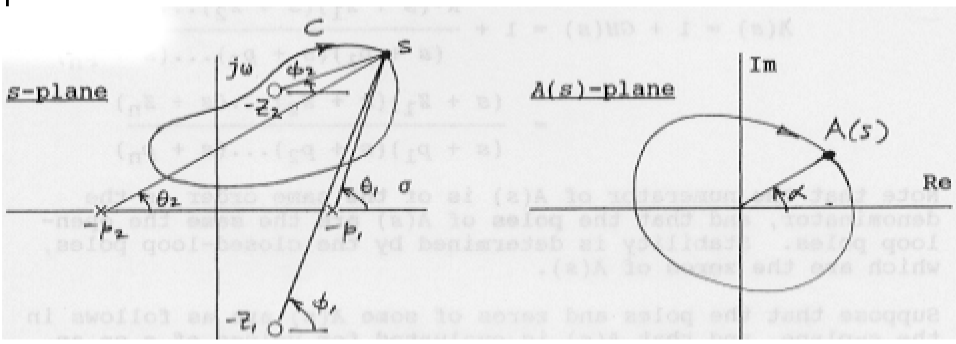
$$\alpha=\phi_1+\phi_2-\theta_1-\theta_2$$

- when there are no singularities within the contour (as above), the phases of the vectors from poles and zeros outside the contour will vary somewhat, but will have a *net change of zero over one complete contour*.

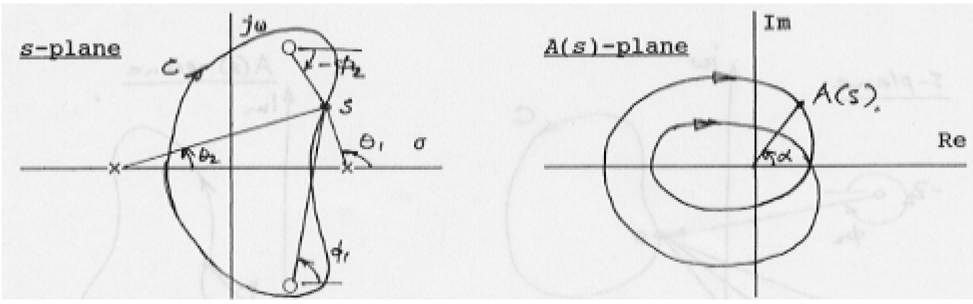
- however, when there are singularities within the contour, the phase from each singularity inside the contour will undergo a net change of 360°, and will induce similar cyclic changes in α. These correspond to encirclements of the origin by the closed curve in the $A(s)$ plane.

- if we travel clockwise along C in s space, zeros of $A(s)$ inside the contour C will lead to clockwise encirclements of the origin; poles will cause anticlockwise encirclements

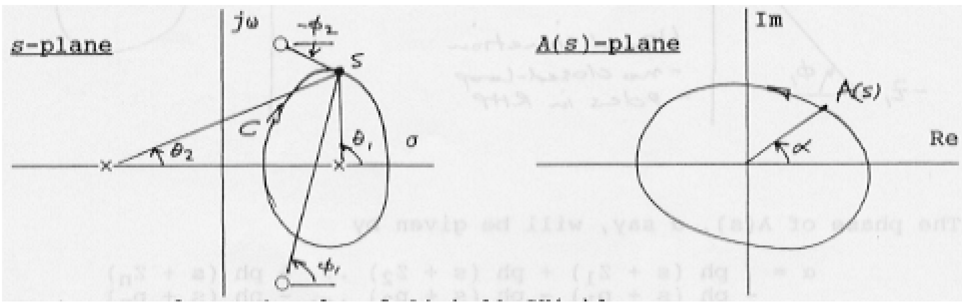
- examples:



  - in this case the contour C encloses 1 zero of $A(s)$, thus ϕ_2 goes through -360^0 (clockwise) and $\theta_1, \theta_2, \phi_1$ go through 0^0. α therefore goes through -360^0.

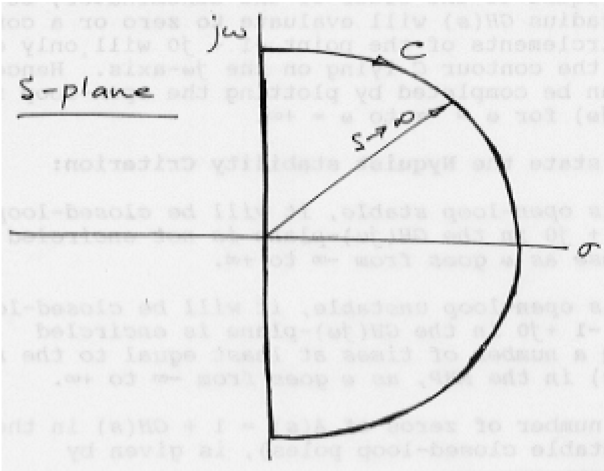


  - here the contour C encloses 2 zeros of $A(s)$, thus ϕ_1 and ϕ_2 each go through -360^0 (clockwise) and θ_1, θ_2 go through 0^0. α therefore goes through -720^0.



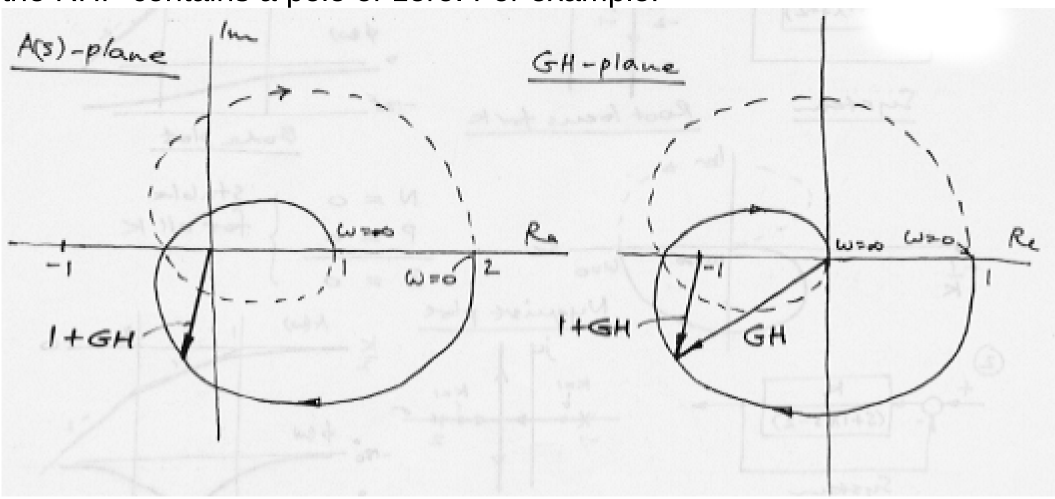
  - here the contour C encloses 1 pole of $A(s)$, thus θ_1 goes through -360^0 (clockwise) and ϕ_1, ϕ_2, θ_2 go through 0^0. α therefore goes through 360^0.

- now, we are interested in the number of **zeros** of $A(s)$ in the RHP, since these determine the closed loop stability of our system. Hence we make the contour C encircle the entire RHP:



- if the RHP contains a pole or zero of $A(s)=1+GH(s)$ then the evaluation of $A(s)$ will encircle the origin.

- alternatively, since $A(s)=1+GH(s)$, the plot of the OLTF $GH(s)$ is simply that of $A(s)$ shifted to the left by 1, the evaluation of *GH*(*s*) will encircle the point $-1+i0$ if the RHP contains a pole or zero. For example:



- as we know, any real system must have the order of the numerator of $GH(s)$ less than or equal to the order of the denominator.

- consider the evaluation of $GH(s)$ at some point on an arc of infinite radius:

$$\left[GH(s)\right]_{s\to\infty}=\left[\frac{K(s+z_1)(s+z_2)\ldots(s+z_m)}{(s+p_1)(s+p_2)\ldots(s+p_n)}\right]_{s\to\infty} \qquad (19.5)$$

1. Because *all* poles and zeros are infinitely far from all points on this arc, the phase angles from *all* these poles and zeros will be the same.
2. all terms in the numerator $|s+z_m|$ will tend to ∞ and those in the denominator $|s+p_n|$ will tend to 0

- both the magnitude and phase of $GH(s)$ are therefore dependent on the relative order of the numerator and denominator (ie. the relative number of OLP's than OLZ's):

$$\left[GH(s)\right]_{s\to\infty} = \left[\frac{K(s+z_1)(s+z_2)\ldots(s+z_m)}{(s+p_1)(s+p_2)\ldots(s+p_n)}\right]_{s\to\infty} \sim \left|s^{m-n}\right|_{s\to\infty} e^{i(m-n)\theta} \qquad (19.6)$$

- when the order of the numerator is less than that of the denominator (ie. there are more OLP's than OLZ's):

$$\left[GH(s)\right]_{s\to\infty} \sim \left|\frac{1}{s^{n-m}}\right|_{s\to\infty} e^{i(m-n)\theta} = 0 \qquad (19.7)$$

- ie. an arc of infinite radius in the s plane maps onto the origin in the $GH(s)$ plane

- when the order of the numerator is the same as that of the denominator (ie. there are the same number of OLP's and OLZ's):

$$\left[GH(s)\right]_{s\to\infty} \sim \left|s^0\right|_{s\to\infty} e^0 = 1 \qquad (19.8)$$

- ie. an arc of infinite radius in the s plane maps onto the point 1 in the $GH(s)$ plane

- thus, depending on the relative order of the numerator and denominator, $GH(s)$ will evaluate to either zero or a constant on the arc of infinite radius. *Thus, any encirclements of the point $-1+i0$ will only occur for that part of the contour C lying on the $i\omega$ axis <u>ie. we can use the OLFR</u>.*

$\Rightarrow$ hence the evaluation can be completed by plotting the open-loop frequency response $GH(i\omega)$ for $\omega \to -\infty$ to $\omega \to +\infty$.

- we can now state the **Nyquist stability Criterion:**

If a system is open-loop stable, it will be closed-loop stable if the point $-1+i0$ in the $GH(i\omega)$ plane is not encircled in a clockwise sense for $\omega \to -\infty$ to *$\omega \to +\infty$.*

If a system is open-loop unstable, it will be closed-loop stable if the point $-1+i0$ in the $GH(i\omega)$ plane is encircled <u>anticlockwise</u> a number of times at least equal to the number of poles of $GH(i\omega)$ in the RHP for $\omega \to -\infty$ to *$\omega \to +\infty$.*

- that is, the number of zeros of $A(s) = 1 + GH(s)$ in the RHP (the number of unstable closed-loop poles), is given by

$$Z = N + P \qquad (19.9)$$

- where $N =$ the number of **clockwise** encirclements of $-1+i0$ and $P =$ the number of poles of $GH(s)$ in the RHP.