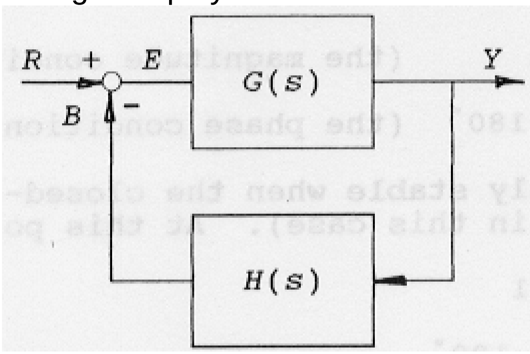
Lecture 18: Frequency-Domain Tests of stability

- consider the standard single-loop system:



- we can determine the stability of the closed-loop system by evaluating the frequency response of the **open-loop transfer function** $GH(i\omega)$. This can be done by using the following means:

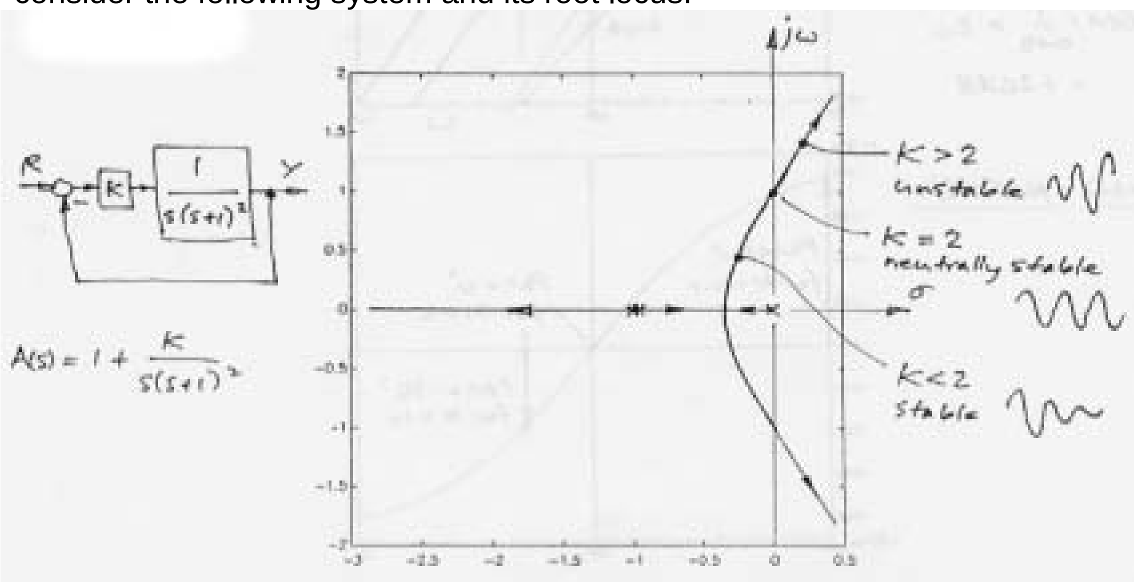
1. Bode plot: the frequency response in terms of the magnitude and phase
2. Nichols chart: the Log-magnitude vs the phase
3. Nyquist plot: a polar plot of the complex number $GH(i\omega)$.

- for stability, the CLTF $Y(s)/R(s)$ must have no **poles** in the right-half of the *s*-plane). That is, the characteristic polynomial $A(s)=1+GH(s)$ must have no **zeros** in the RHP.

Example

- earlier, we used the root locus technique to investigate the variation of the zeros of $A(s)$ with (typically) the loop gain, when $GH(s)$ was available in factored form.

- consider the following system and its root locus:



- the characteristic equation for this system is:

$$A(s) = 1 + GH(s) = 1 + \frac{K}{s(s+1)^2} = 0 \tag{18.1}$$

- the root locus shows the variation of the zeros of $A(s)$ as K varies from 0 to ∞. All points on the locus satisfy the following conditions:

  the magnitude condition: $|GH(s)| = 1$

  the phase condition: $ph[GH(s)] = (2k-1)180^0,\ k = 0, \pm 1, \pm 2, \ldots$

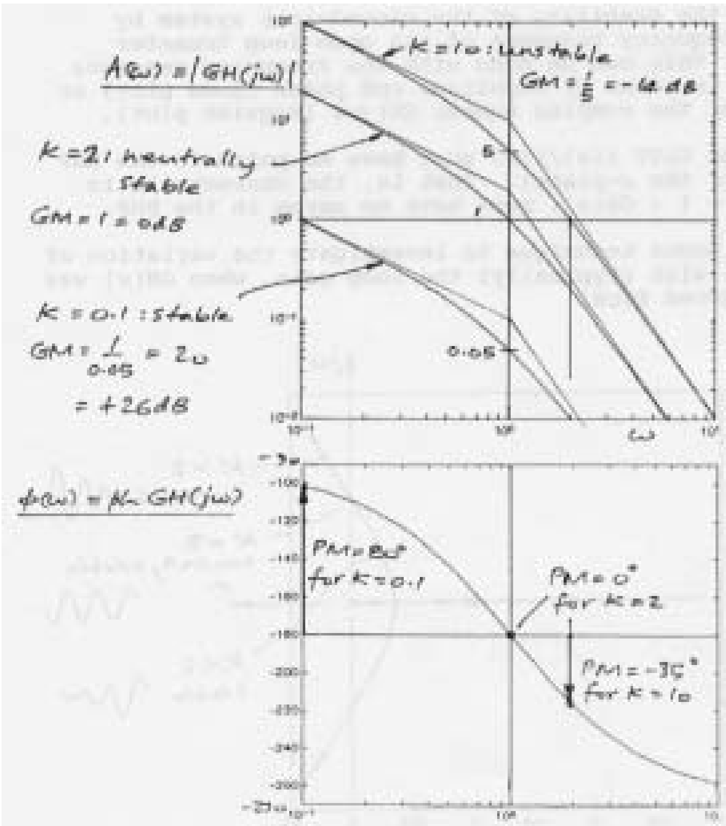
18.1 The gain and phase margins

- the system is **neutrally stable** when the closed-loop roots are at $s = \pm i\omega$ (when $K = 2$ in this case). At this point we have

$$\begin{aligned} |GH(i\omega)| &= 1 \\ ph[GH(i\omega)] &= -180^0 \end{aligned} \tag{18.2}$$

- whether these conditions for neutral stability are met can be determined from the Bode plot of the OLTF $GH(i\omega)$.

- for our example, $GH(s) = K/\left[s(s+1)^2\right]$:



- as is the case in many systems, the above example shows that increasing gain leads to instability, whilst reducing the gain tends to stabilise the system. Note that this is not always true, however.

- in general, the system will be stable if $|GH(i\omega)| < 1$ at the frequency for which $ph[GH(i\omega)] = -180^0$. For such cases, the degree of **relative stability** can be expressed by the **Gain Margin (GM)** and **Phase Margin (PM)**:

*The **gain margin** indicates the amount by which the gain may be increased before the system becomes unstable:*

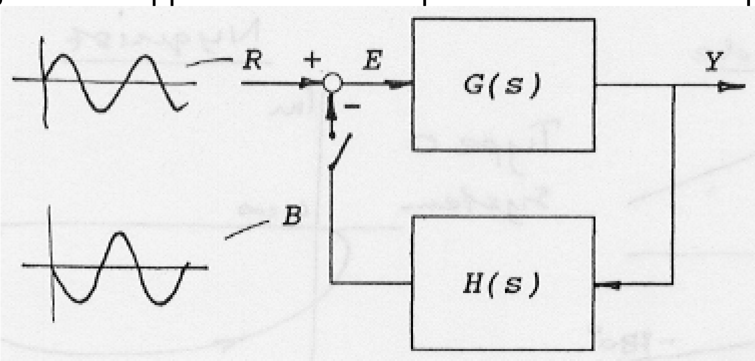
$$GM = 1/|GH(i\omega_1)| \tag{18.3}$$

- where ω_1 is the **phase crossover frequency**: $ph[GH(i\omega_1)] = -180^0$

*The **phase margin** indicates the additional phase lag which can be tolerated in the OLTF at the **gain crossover frequency** ω_c before the system becomes unstable:*

$$PM = 180^0 + ph[GH(i\omega_c)] \tag{18.4}$$

- a qualitative understanding of this stability condition can be gained from the following argument. Suppose a switch is opened in the feedback path:



1. suppose that the reference input R is sinusoidal, and the frequency is adjusted until the feedback signal in the open circuit lags the input by exactly 180°.

2. then, if the switch is closed, the negative feedback signal will be in phase (360° phase "difference") with the reference input, which may now be removed so that all excitation of the system will come from its own feedback.

3. if the open-loop gain $|GH(i\omega_1)|$ at this frequency is less than 1, the oscillations will die out; if it exceeds unity they will grow. If $|GH(i\omega_1)| = 1$, the system will exhibit a sustained oscillation at this frequency, the phase crossover frequency ω_1; i.e. it will be neutrally stable.

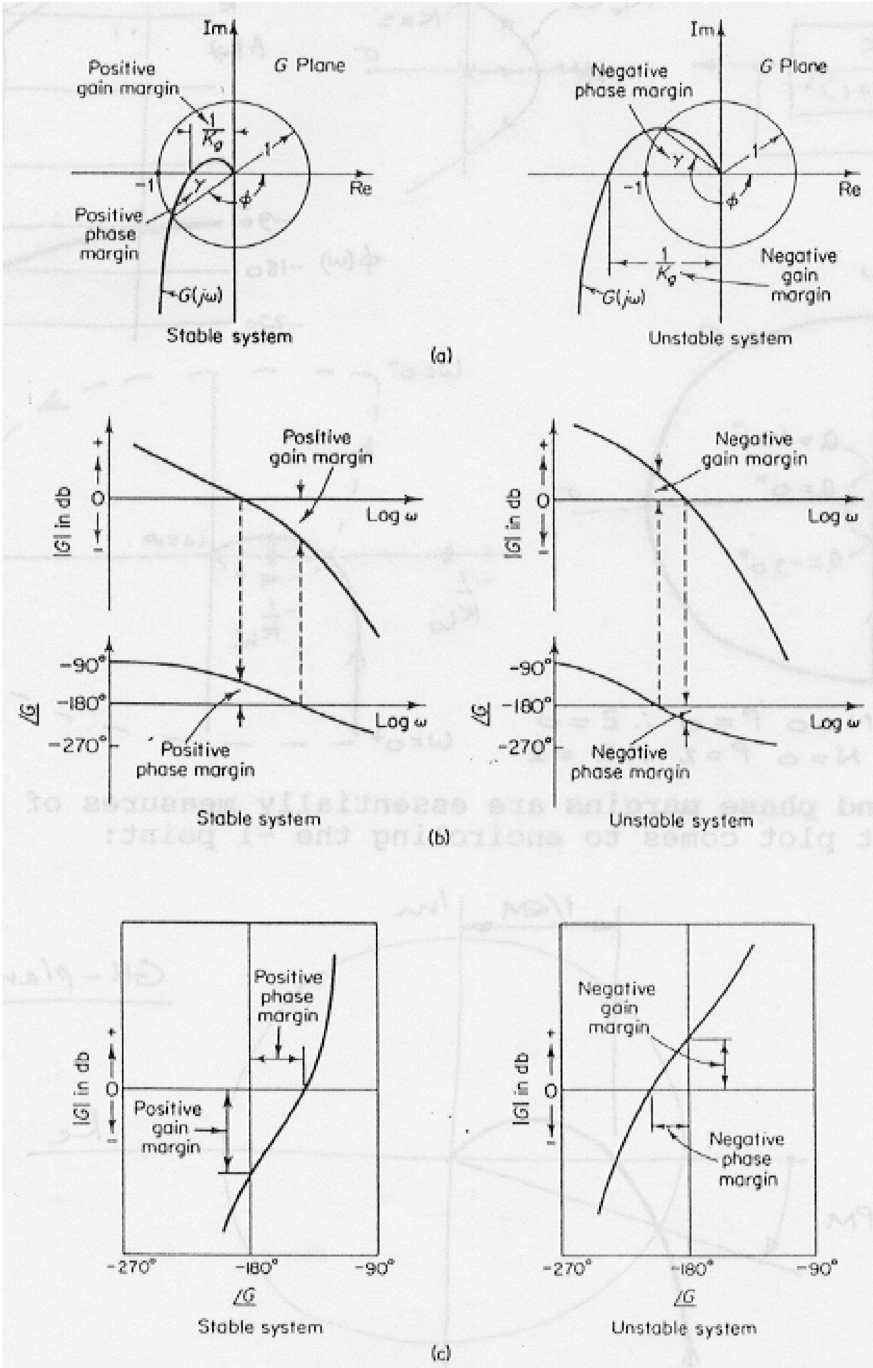
18.2 Nichols and Nyquist charts

- another way to display the open-loop frequency response is to plot the Bode diagram coordinates against each other. A plot of the Log-magnitude $Lm = \log\left|GH(i\omega)\right|$ vs the phase $\varphi(\omega) = ph\left[GH(i\omega)\right]$ is known as a **Nichols chart**:

Nyquist (polar)

Bode

Nichols



- a polar plot of the complex number GH is termed the **Nyquist plot**.