Lecture 17: Procedure for plotting root loci

- in lecture 16 we considered the characteristic equation of a closed loop transfer function with the form
  \[ 1 + K \frac{B(s)}{A(s)} = 0 \]  
  \[ (17.1) \]

- where \( K \) = the parameter of interest for defining the locus and \( A(s) \) and \( B(s) \) are factored polynomials:
  \[ B(s) = (s + z_1)(s + z_2)(s + z_3) \ldots (s + z_m) \]
  \[ A(s) = (s + p_1)(s + p_2)(s + p_3) \ldots (s + p_n) \]  
  \[ (17.2) \]

- ie. there are \( m \) open loop zeros and \( n \) open loop poles

- therefore a simple definition of the root locus is
  
  The root locus is the set of values of \( s \) for which \( 1 + K \frac{B(s)}{A(s)} = 0 \) is satisfied as the real parameter \( K \) varies from 0 to \( +\infty \).

- in lecture 16 we established two conditions in order for values of \( s \) to satisfy the characteristic equation:
  
  magnitude condition
  \[ \left| \frac{B(s)}{A(s)} \right| = \frac{1}{K} \]
  
  phase condition
  \[ \text{ph} \left( \frac{B(s)}{A(s)} \right) = (2k - 1) \cdot 180° \text{ for } k = 0, \pm 1, \pm 2, \ldots \]  
  \[ (17.3) \]

- equivalently, the phase condition in equation (17.3) can instead be written as
  \[ \sum_{i=1}^{m} \text{ph}(s + z_i) - \sum_{i=1}^{n} \text{ph}(s + p_i) = (2k - 1) \cdot 180° \]  
  \[ (17.4) \]

- with the conditions (17.3) determining which values of \( s \) belong to the characteristic equation, we can develop general rules for plotting the root loci of arbitrary systems.

17.1 Procedure for plotting root loci

1. Plot the open-loop poles (X) and zeros (0) on the s-plane

2. If the characteristic polynomial is order \( n \), there will be \( n \) closed-loop poles and hence \( n \) branches of the root locus.

3. The branches of the locus begin at the open-loop poles and terminate at the open-loop zeros.

- this can be seen by rewriting equation (17.1) as
\[ A(s) + K.B(s) = 0 \]  \hspace{1cm} (17.5)

- thus, if \( K = 0 \)
  \[ A(s) = 0 \Rightarrow s = -p_i, \ i = 1, 2, ..., n \]  \hspace{1cm} (17.6)
- i.e. the open loop poles

- and if \( K \rightarrow \infty \)
  \[ K \rightarrow \infty \Rightarrow B(s) = 0 \Rightarrow s = -z_i, \ i = 1, 2, ..., m \]  \hspace{1cm} (17.7)
- i.e. the open loop zeroes

- note that if there are more branches than open-loop zeros (i.e. \( n > m \)), then \( n - m \) branches terminate at \( s \rightarrow \infty \).

4. The locus lies on the real axis whenever there is an odd number of open-loop poles and zeros to the right.
- this follows simply from the phase condition

5. The root loci are symmetrical about the real axis.
- complex roots appear as conjugate pairs.

6. Branches that terminate at infinity do so asymptotically to lines oriented at angles:
  \[ \theta_k = \frac{(2k-1)180^\circ}{(n-m)} \quad k = 0, 1, 2, ..., (n - m - 1) \]  \hspace{1cm} (17.8)
- the asymptotes intersect the real axis at the ‘centroid’ of the open-loop poles (regarded as having mass = +1) and zeros (with mass = -1); i.e., at
  \[
  s = \frac{\sum \text{poles} - \sum \text{zeros}}{(n - m)} = \frac{\sum (-p_i) - \sum (-z_i)}{(n - m)}
  \]  \hspace{1cm} (17.9)
- for sufficiently large \( s \), the phases of the vectors from all the open-loop poles and zeros are be approximately equal. In the phase condition summation (17.4), the phases of the vectors from the \( m \) poles will thus be cancelled by the phases of the vectors from the zeros.
- the remaining n-m vectors (each with phase \( \vartheta_k \)) must satisfy
  \[
  (n-m) \vartheta_k = (2k-1)180^\circ
  \] (17.10)
- see the textbook for a proof of centroid formula.

7. Branches of the locus leave the open-loop poles at departure angles given by

\[
\vartheta_{\text{dep}} = (2k-1)180^\circ - \sum(\text{angles from other poles}) + \sum(\text{angles from zeros}) = (2k-1)180^\circ - \sum_{i=1}^{n-1} \vartheta_i + \sum_{i=1}^{m} \varphi_i
\] (17.11)

- this condition follows from the phase condition.

8. Branches of the locus reach the open-loop zeros at arrival angles given by

\[
\varphi_{\text{arr}} = (2k-1)180^\circ - \sum(\text{angles from other poles}) + \sum(\text{angles from zeros}) = (2k-1)180^\circ + \sum_{i=1}^{n} \vartheta_i - \sum_{i=1}^{m-1} \varphi_i
\] (17.12)

9. Multiple roots occur at “breakaway” or “break-in” points on the real axis, or in complex conjugate pairs, where

\[
\frac{dK}{ds} = 0
\] (17.13)

- i.e. at values of \( s \) for which

\[
A(s)\frac{dB(s)}{ds} - B(s)\frac{dA(s)}{ds} = 0
\] (17.14)

- this can be seen by differentiating equation (17.1):

\[
\frac{d}{ds} \left[ \frac{K.B(s)}{A(s)} \right] = 0
\]

\[
\Rightarrow K.B(s)\frac{dK}{ds} + K \left[ \frac{1}{A(s)} \frac{dB(s)}{ds} + B(s) \frac{d}{ds} \left( \frac{1}{A(s)} \right) \right] = 0
\] (17.15)

\[
\Rightarrow A(s) \frac{dB(s)}{ds} - B(s) \frac{dA(s)}{ds} = 0
\]

- for example, moving along the real axis from an open-loop pole towards a breakaway point, \( K \) increases until it reaches a maximum at the breakaway point and then decreases as the second open-loop pole is approached. Similarly \( K \) increases from a minimum at a break-in point along each branch approaching an open-loop zero on the real axis. In general, \( K \) will have a stationary value where there is a multiple root.
10. At any multiple root, the tangents to the branches of the locus divide the surrounding space into sectors of included angle $180°/q$, where $q$ is the order of the root. Branches enter and leave the multiple root location alternately.

11. Values of $K$ for which the branches cross the imaginary axis can be found by applying the Routh stability criterion. The characteristic equation obtained by substituting $s = i\omega$ and the critical values of $K$ may be solved for the $i\omega$ axis crossover frequencies.

12. Determining the general shape of the root locus may be assisted by noting that, provided $m \leq n - 2$, the sum of the closed-loop roots is constant and independent of $K$. Hence, if some of the roots move to the left as $K$ is increased, others must move to the right to conserve the sum of the roots.

- this can be seen from once again considering the characteristic equation, where we label the roots of the characteristic equation $r_i$
  \[ A(s) + KB(s) = 0 \]
  \[ \Rightarrow \prod_{i=1}^{n} (s + p_i) + K \prod_{i=1}^{m} (s + z_i) = \prod_{i=1}^{n} (s + r_i) \]  \[
(17.16)
\]
- where $p_i, z_i$ are the open loop poles and zeros respectively and $r_i$ are the roots of the characteristic equation (i.e. the closed loop poles)

- from equation (17.16), it follows that
  \[
(s^n + \sum p_i s^{n-1} + \ldots) + K (s^m + \sum z_i s^{m-1} + \ldots) = s^n + \sum r_i s^{n-1} + \ldots
\]
  \[
(17.17)
\]
- if $m < n - 1$, equation (17.17) shows that the term $Ks^m$ has lower order than the terms $\sum p_i s^{n-1}$ and $\sum r_i s^{n-1}$. Thus
  \[
\sum p_i = \sum r_i
\]
  \[
(17.18)
\]
- if $m < n - 1$
now equation (17.18) is independent of $K$, so the sum of the close loop poles is constant and equal to the sum of the open loop poles if $m < n - 1$.

13. The gain $K$ for a specific point on the locus, $s = s_1$, may be calculated from the magnitude condition

$$K_{s=s_1} = \frac{|A(s_1)|}{|B(s_1)|} = \frac{|s_1 + p_1||s_1 + p_2|...}{|s_1 + z_1||s_1 + z_2|...} \tag{17.19}$$

14. Finding roots for a given value of $K$ usually follows after having determined $K$ such that the dominant closed-loop poles have desirable values. The location of the remaining roots may be found by trial-and-error application of the magnitude condition, or dividing the characteristic equation by the known roots and factoring the remainder.