Lecture 17: Procedure for plotting root loci

- in lecture 16 we considered the characteristic equation of a closed loop transfer function with the form

$$1 + K.B(s)/A(s) = 0 \tag{17.1}$$

- where $K =$ the parameter of interest for defining the locus and $A(s)$ and $B(s)$ are factored polynomials:

$$\begin{aligned} B(s) &= (s+z_1)(s+z_2)(s+z_3)\ldots(s+z_m) \\ A(s) &= (s+p_1)(s+p_2)(s+p_3)\ldots(s+p_n) \end{aligned} \tag{17.2}$$

- ie. there are m open loop zeros and n open loop poles

- therefore a simple definition of the root locus is

  The root locus is the set of values of s *for which* $1+K.B(s)/A(s)=0$ *is satisfied as the real parameter* K *varies from* 0 *to* $+\infty$.

- in lecture 16 we established two conditions in order for values of s to satisfy the characteristic equation:

$$\begin{aligned} &\textit{magnitude condition} \qquad \left|\frac{B(s)}{A(s)}\right| = \frac{1}{K} \\ &\textit{phase condition} \qquad ph\left(\frac{B(s)}{A(s)}\right) = (2k-1).180^\circ \;\; \textit{for} \;\; k = 0, \pm 1, \pm 2, \ldots \end{aligned} \tag{17.3}$$

- equivalently, the phase condition in equation (17.3) can instead be written as

$$\sum_{i=1}^{m} ph(s+z_i) - \sum_{i=1}^{n} ph(s+p_1) = (2k-1).180^\circ \tag{17.4}$$

- with the conditions (17.3) determining which values of s belong to the characteristic equation, we can develop general rules for plotting the root loci of arbitrary systems.

17.1 Procedure for plotting root loci

1. Plot the open-loop poles (X) and zeros (0) on the s-plane

2. If the characteristic polynomial is order n, there will be n closed-loop poles and hence n branches of the root locus.

3. The branches of the locus begin at the open-loop poles and terminate at the open-loop zeros.

   - this can be seen by rewriting equation (17.1) as

$$A(s)+K.B(s)=0 \qquad (17.5)$$

- thus, if $K=0$

$$A(s)=0 \Rightarrow s=-p_i,\ i=1,2,...,n \qquad (17.6)$$

- ie. the open loop poles

- and if $K \to \infty$

$$K \to \infty \Rightarrow B(s)=0 \Rightarrow s=-z_i,\ i=1,2,...,m \qquad (17.7)$$

- ie. the open loop zeroes

- note that it there are more branches than open-loop zeros (ie. $n>m$), then $n-m$ branches terminate at $s \to \infty$.

4. The locus lies on the real axis whenever there is an odd number of open-loop poles and zeros to the right.

   - this follows simply from the phase condition

5. The root loci are symmetrical about the real axis.

   - complex roots appear as conjugate pairs.

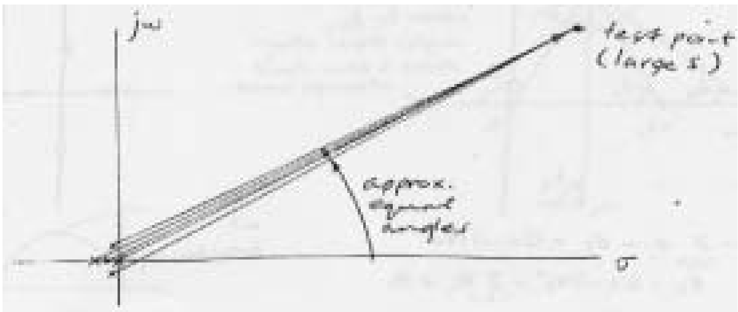
6. Branches that terminate at infinity do so asymptotically to lines oriented at angles:

$$\theta_k = \frac{(2k-1)180°}{(n-m)} \quad k=0,1,2,...,(n-m-1) \qquad (17.8)$$

   - the asymptotes intersect the real axis at the ‘centroid’ of the open-loop poles (regarded as having mass = +1) and zeros (with mass = -1); i.e., at

$$s=\frac{\sum poles-\sum zeros}{(n-m)} = \frac{\sum(-p_i)-\sum(-z_i)}{(n-m)} \qquad (17.9)$$

   - for sufficiently large s, the phases of the vectors from all the open-loop poles and zeros are be approximately equal. In the phase condition summation (17.4), the phases of the vectors from the m poles will thus be cancelled by the phases of the vectors from the zeros.



- the remaining n-m vectors (each with phase θ_k) must satisfy

$$(n-m)\theta_k = (2k-1)180° \quad (17.10)$$

- see the textbook for a proof of centroid formula.

7. Branches of the locus leave the open-loop poles at departure angles given by

$$\begin{aligned}\theta_{dep} &= (2k-1).180° - \Sigma(angles\ \ from\ other\ \ poles) + \Sigma(angles\ \ from\ zeros)\\ &= (2k-1).180° - \sum_{i=1}^{n-1}\theta_i + \sum_{i=1}^{m}\varphi_i\end{aligned} \quad (17.11)$$

- this condition follows from the phase condition.

8. Branches of the locus reach the open-loop zeros at arrival angles given by

$$\begin{aligned}\varphi_{arr} &= (2k-1).180° - \Sigma(angles\ \ from\ other\ \ poles) + \Sigma(angles\ \ from\ zeros)\\ &= (2k-1).180° + \sum_{i=1}^{n}\theta_i - \sum_{i=1}^{m-1}\varphi_i\end{aligned} \quad (17.12)$$

9. Multiple roots occur at "breakaway" or "break-in" points on the real axis, or in complex conjugate pairs, where

$$\frac{dK}{ds} = 0 \quad (17.13)$$

- i.e. at values of s for which

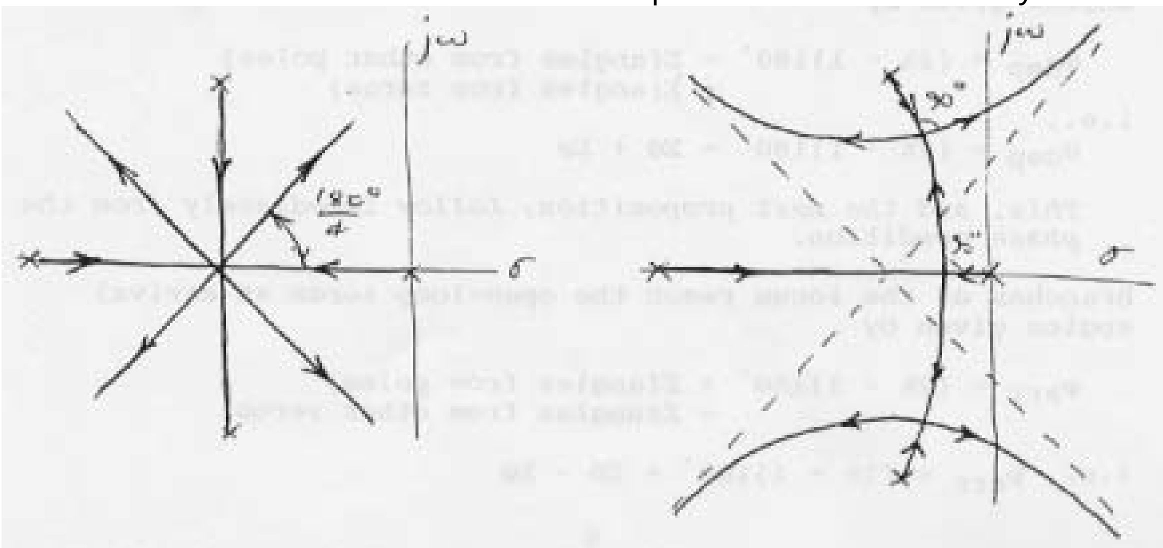
$$A(s)\frac{dB(s)}{ds} - B(s)\frac{dA(s)}{ds} = 0 \quad (17.14)$$

- this can be seen by differentiating equation (17.1):

$$\begin{aligned}&\frac{d}{ds}\left[\frac{K.B(s)}{A(s)}\right] = 0\\ &\Rightarrow \frac{K.B(s)}{A(s)}\underbrace{\frac{dK}{ds}}_{=0} + K\underbrace{\left[\frac{1}{A(s)}\frac{dB(s)}{ds} + B(s)\frac{d}{ds}\left(\frac{1}{A(s)}\right)\right]}_{d[B(s)/A(s)]/ds} = 0 \quad (17.15)\\ &\Rightarrow A(s)\frac{dB(s)}{ds} - B(s)\frac{dA(s)}{ds} = 0\end{aligned}$$

- for example, moving along the real axis from an open-loop pole towards a breakaway point, K increases until it reaches a maximum at the breakaway point and then decreases as the second open-loop pole is approached. Similarly K increases from a minimum at a break-in point along each branch approaching an open-loop zero on the real axis. In general, K will have a stationary value where there is a multiple root.

10. At any multiple root, the tangents to the branches of the locus divide the surrounding space into sectors of included angle $180^0/q$, where q is the order of the root. Branches enter and leave the multiple root location alternately.



11. Values of K for which the branches cross the imaginary axis can be found by applying the Routh stability criterion. The characteristic equation obtained by substituting $s = i\omega$ and the critical values of K may be solved for the $i\omega$ axis crossover frequencies.

12. Determining the general shape of the root locus may be assisted by noting that, provided $m \le n-2$, the sum of the closed-loop roots is constant and independent of K. Hence, if some of the roots move to the left as K is increased, others must move to the right to conserve the sum of the roots.

- this can be seen from once again considering the characteristic equation, where we label the roots of the characteristic equation r_i

$$A(s) + KB(s) = 0$$
$$\Rightarrow \prod_{i=1}^{n}(s+p_i) + K\prod_{i=1}^{m}(s+z_i) = \prod_{i=1}^{n}(s+r_i) \tag{17.16}$$

- where p_i, z_i are the open loop poles and zeros respectively and r_i are the roots of the characteristic equation (ie. the *closed* loop poles)

- from equation (17.16), it follows that

$$\left(s^n + \sum p_i s^{n-1} + ...\right) + K\left(s^m + \sum z_i s^{m-1} + ...\right) = s^n + \sum r_i s^{n-1} + ... \tag{17.17}$$

- if $m < n-1$, equation (17.17) shows that the term Ks^m has lower order than the terms $\sum p_i s^{n-1}$ and $\sum r_i s^{n-1}$. Thus

$$\sum p_i = \sum r_i \tag{17.18}$$

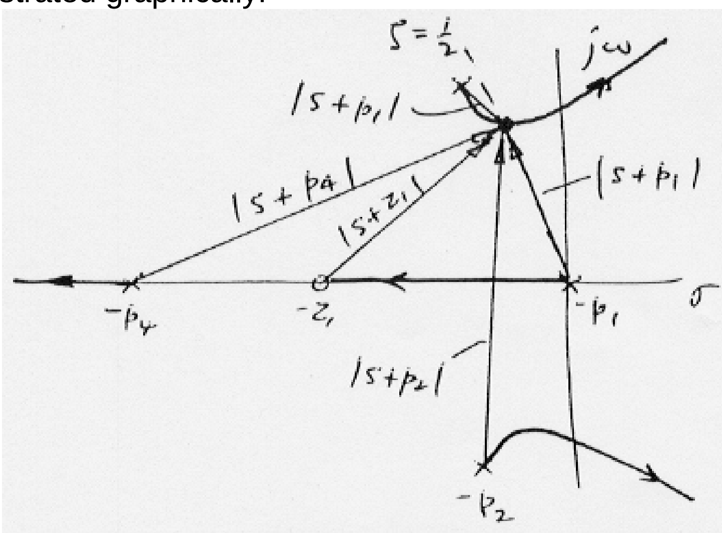
- if $m < n-1$

- now equation (17.18) is independent of K, so the sum of the close loop poles is constant and equal to the sum of the open loop poles if $m < n-1$

13. The gain K for a specific point on the locus, $s = s_1$, may be calculated from the magnitude condition

$$K_{s=s_1} = \left|\frac{A(s_1)}{B(s_1)}\right| = \frac{|s_1 + p_1||s_1 + p_2|...}{|s_1 + z_1||s_1 + z_2|...} \tag{17.19}$$

- This is illustrated graphically:



14. Finding roots for a given value of K usually follows after having determined K such that the dominant closed-loop poles have desirable values. The location of the remaining roots may be found by trial-and-error application of the magnitude condition, or dividing the characteristic equation by the known roots and factoring the remainder.