Lecture 16: Evans' Root Locus Method

- Evans (1948) developed a method for plotting the variation of the roots of a polynomial equation as the value of a coefficient changes.

- the usual application of the method in control design is to plot the variation in the s-plane of the closed-loop poles as a system parameter (usually the open-loop gain) varies from zero to infinity.

- we therefore get a complete solution of the system characteristic equation for all values of the gain (i.e., a locus of the roots of the equation).

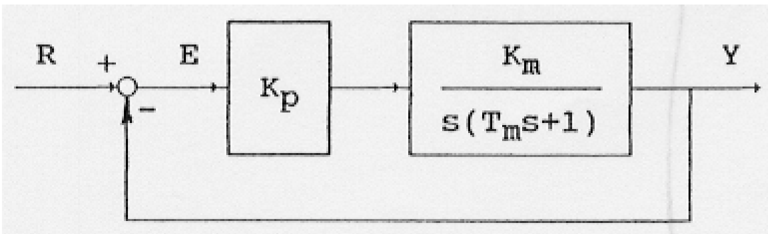
- recall that the location of the poles in the s-plane can be correlated with the nature of the closed-loop transient responses. If a suitable set of closed-loop poles can be identified, we can therefore select the corresponding gain from the locus. If not, a *dynamic compensator* can be designed to re-shape the locus to give a more favourable choice of closed-look poles.

- even though solutions of the characteristic equation can be routinely obtained by numerical means, the root locus method remains a powerful *design* tool, because it gives insight into the dynamic characteristics of the system, including its sensitivity to parameter changes, and provides a basis for *synthesizing* compensators.

- quite accurate root loci can be *sketched* rapidly, which is also an aid in experimenting with design solutions.

16.1 A simple example of a root locus plot

- consider a position control system with a DC motor as actuator and employing a 'proportional' control algorithm (as in lecture 10):



- the open-loop transfer function (OLTF) is

$$G(s) = \underbrace{\frac{K_p K_m}{s(T_m s + 1)}}_{Bode\ form} = \underbrace{\frac{K}{s(s+p)}}_{Evans\ form} \tag{16.1}$$

- when the numerator and denominator polynomials or factors in the transfer function are written so that the coefficient of the highest power of s is unity, we shall say it is in "Evans form" (to distinguish it from the Bode form).

- the open-loop Evans gain is $K = K_p K_m / T_m$, and the open-loop poles are at $s = 0$ *and* $s = -p = -1/T_m$.

- the closed-loop transfer function (CLTF) is

$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)} \qquad (16.2)$$

- and the characteristic equation for the closed-loop system is

$$1 + \frac{K}{s(s+p)} = 0$$
$$\Rightarrow s^2 + ps + K = 0 \qquad (16.3)$$

- the roots of the characteristic equation are the closed-loop poles:

$$s_{1,2} = -p/2 \pm \sqrt{\left[\left(p/2\right)^2 - K\right]} \qquad (16.4)$$

- from which the following can be determined:

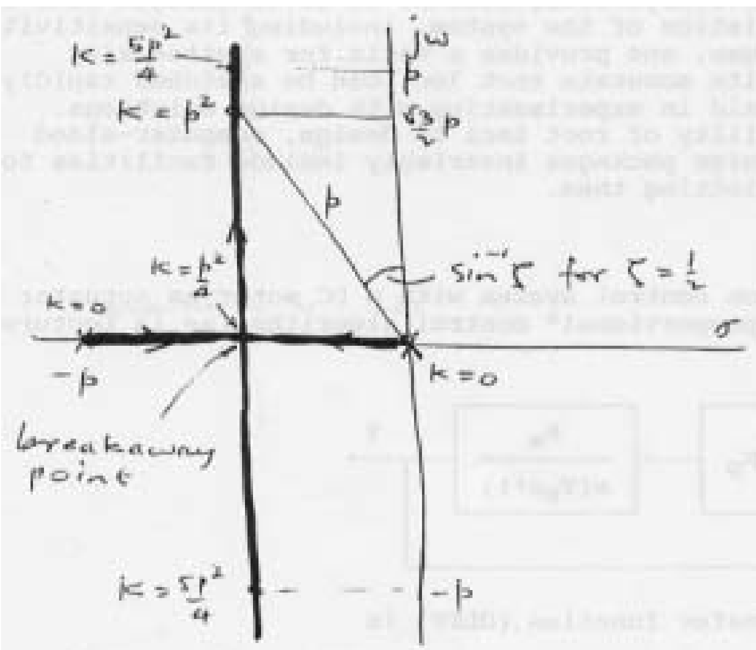
1. $s_{1,2} = 0, -p$ when $K = 0$

2. $s_{1,2} = -p/2$ when $K = \left(p/2\right)^2$

3. the roots are real for $0 \le K \le \left(p/2\right)^2$ and complex for $K > \left(p/2\right)^2$.

4. when the roots are complex, the real part is independent of K. Thus the locus for this portion of the graph will be vertical lines extending to $\pm\infty$

- these are sketched as



- thus, when $0 \le K \le (p/2)^2$, the natural response of the system will consist only of damped exponential behaviour. However, once $K > (p/2)^2$, damped oscillatory behaviour will occur.

- note that as the system is of second-order there are two roots, so that there are two *branches* of the root locus. The branches "begin" at the open-loop poles where $K = 0$.

16.2 Evans' method

- let us write the characteristic equation of the close loop transfer function in the following form

$$1 + K.B(s)/A(s) = 0 \tag{16.5}$$

- where $A(s)$ and $B(s)$ are factored polynomials:

$$\begin{aligned} B(s) &= (s+z_1)(s+z_2)(s+z_3)...(s+z_m) \\ A(s) &= (s+p_1)(s+p_2)(s+p_3)...(s+p_n) \end{aligned} \tag{16.6}$$

- and $K =$ the parameter of interest for defining the locus, where $0 \le K \le \infty$. Then equation (16.5) gives

$$\frac{B(s)}{A(s)} = \left|\frac{B(s)}{A(s)}\right| e^{ph(B/A)} = -\frac{1}{K} = \frac{1}{K} e^{i(2k-1)\pi} \tag{16.7}$$

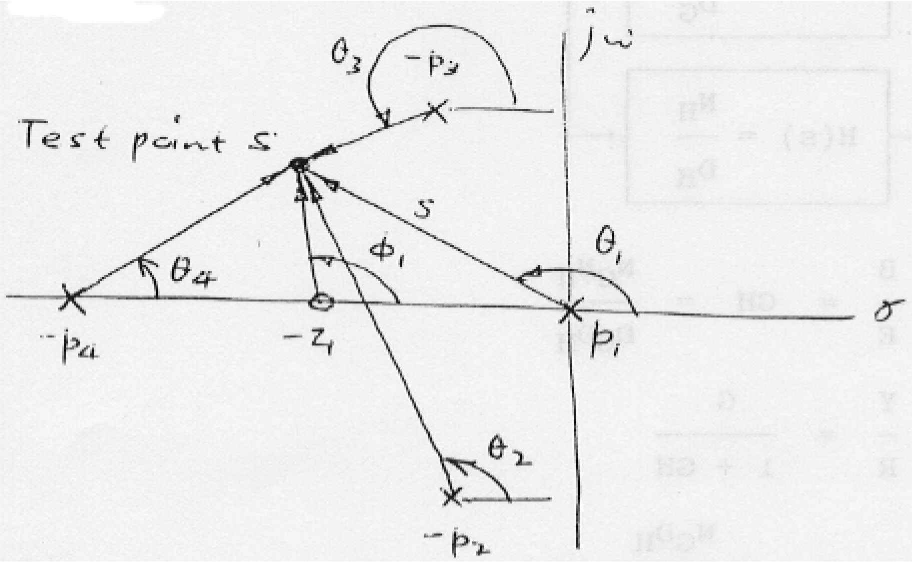
- ie. there are two conditions in order for values of s to satisfy the characteristic equation:

$$\begin{aligned} &\textit{magnitude condition} \quad \left|\frac{B(s)}{A(s)}\right| = \frac{1}{K} \\ &\textit{phase condition} \quad ph\left(\frac{B(s)}{A(s)}\right) = (2k-1).180^\circ \;\; \textit{for} \;\; k = 0, \pm 1, \pm 2, ... \end{aligned} \tag{16.8}$$

- since the phase of the overall transfer function is simply the sum of the phases of the more simple transfer functions contained within it, it follows that

$$\sum_{i=1}^{m} \underbrace{ph(s+z_i)}_{\equiv \varphi_i} - \sum_{i=1}^{n} \underbrace{ph(s+p_1)}_{\equiv \theta_i} = (2k-1).180^\circ \tag{16.9}$$

- which can be sketched as



- *note that the phase condition is sufficient to define the locus as K varies from 0 to ∞.*

- the magnitude condition allows calibration of the locus with specific values of K.