Lecture 15: Stability

- a system is generally regarded as stable if its response to a given input is bounded. Note that there are many subtleties in the precise definitions of stability for different types of system, subjected to different kinds of input, and with different variables considered as the response.

- for nonlinear systems, many different definitions of stability are employed to take account of the effect of the size and type of input, initial conditions, etc. In this case, stability is a property of a particular time history.

- *for linear, time-invariant systems, however, stability is a property of the system.*

- several definitions of stability can be employed for linear systems, depending on whether we are concerned only with the external, "input-output" behaviour (as revealed by the system transfer function or impulse response function), or if we take account also of the internal state of the system. We shall consider here only external stability:

  A system is bounded input-bounded output (BIBO) stable if the zero-state (ie. zero initial conditions) response to any bounded input is also bounded (regardless of what goes on inside the system).

- note that a function g(t) is bounded if its magnitude does not go to infinity in the time interval [0, ∞);

$$|g(t)| < M < \infty \tag{15.1}$$

- if the function $g(t)$ is taken to be the inverse Laplace transform of an impulse transfer function $G(s)$:

$$g(t) = \mathcal{L}^{-1}\left[G(s)\right] \tag{15.2}$$

- where $G(s) = B(s)/A(s)$, the BIBO stability condition in equation (15.1) will be satisfied if all the poles of $G(s)$ (i.e., the roots of the characteristic equation A(s) = 0) have negative real parts.

  Thus, a necessary and sufficient condition for BIBO stability of a system described by the transfer function $G(s)$ is that all the poles of $G(s)$ lie in the left half of the s-plane.

- note that the zeros of $G(s)$ play no part in the stability.

- it is not always convenient to compute the roots of the characteristic equation to determine whether a system is BIBO stable or not. For example, we may want to establish analytical expressions for the bounds on certain system parameters to ensure stability. We shall examine several tests of stability which do not require explicit solution of the closed-loop characteristic equation:

1. Using coefficients of the characteristic equation (Routh's method).
2. Evan's root locus method.
3. Nyquist's polar plot method.

15.1 Routh's stability criterion

- Routh's stability criterion allows us to check for the existence of roots of the characteristic equation with zero or positive real parts without solving the equation.

- let the characteristic equation of an *n*th-order system be
$$A(s) = a_0 s^n + a_1 s^{n-1} + \ldots + a_n = 0; \;\; a_0 > 0 \qquad (15.3)$$

- if any coefficients are zero or negative, there exists at least one root with a non-negative real part. To illustrate this, consider the factors
$$(s+a)(s^2+bs+c) = s^3 + (a+b)s^2 + (b+c)s + ac \qquad (15.4)$$

- all of *a*, *b* and *c* must be positive if the roots are to have negative real parts; the coefficients in the polynomial will then necessarily be positive.

  Hence, a <u>necessary</u> (but not sufficient) condition for stability is that all the coefficients in the characteristic polynomial be present and positive.

- Routh (1874) (and later Hurwitz in 1895) established <u>necessary and sufficient</u> conditions for stability. These require us to form the <u>Routh array</u>:

$$\begin{array}{ll}
s^n: & a_0 \; a_2 \; a_4 \; a_6 \; \cdots (even) \\
s^{n-1}: & a_1 \; a_3 \; a_5 \; a_7 \; \cdots (odd) \\
s^{n-2}: & b_1 \; b_2 \; b_3 \; b_4 \; \cdots \\
s^{n-3}: & c_1 \; c_2 \; c_3 \; c_4 \; \cdots \\
\ldots & \\
\ldots & \\
s^2: & e_1 \; e_2 \\
s^1: & f_1 \\
s^0: & g_1
\end{array}$$

- where

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1} = \frac{-\begin{vmatrix} a_0 & a_2 \\ a_1 & a_3 \end{vmatrix}}{a_1} \qquad c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1} = \frac{-\begin{vmatrix} a_1 & a_3 \\ b_1 & b_2 \end{vmatrix}}{b_1} \qquad d_1 = \frac{c_1 b_2 - b_1 c_2}{c_1} = \frac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}}{c_1}$$

$$b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1} = \frac{-\begin{vmatrix} a_0 & a_4 \\ a_1 & a_5 \end{vmatrix}}{a_1} \qquad c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1} = \frac{-\begin{vmatrix} a_1 & a_5 \\ b_1 & b_3 \end{vmatrix}}{b_1} \qquad \ldots$$

$$b_3 = \frac{a_1 a_6 - a_0 a_7}{a_1} = \frac{-\begin{vmatrix} a_0 & a_6 \\ a_1 & a_7 \end{vmatrix}}{a_1} \qquad \ldots$$

...

- note that, to simplify the calculations, we can multiply (or divide) the coefficients in an entire row by a positive number without affecting the subsequent stability conclusions.

- <u>Routh's stability criterion:</u>

  The system will be stable if all the elements in the first column of the Routh array are positive (i.e., all of $a_0, a_1, b_1, c_1, \ldots,, f_1, g_1 > 0$*).*

  If not all elements in the first row are positive, the number of roots with positive real parts is then given by the number of sign changes in the first column. (Note that a sequence + - + counts as two sign changes.

- <u>example</u>: The characteristic polynomial of a 6th-order system is

$$A(s) = s^6 + 4s^5 + 3s^4 + 2s^3 + s^2 + 4s + 4 \tag{15.5}$$

- the actual roots are s = -3.2644, 0.6769 ± j0.7488, -0.6046 ± j0.9935, -0.8858, two of which have positive real parts. Note that all the polynomial coefficients are present and positive. Forming the Routh array:

$$\begin{array}{lllll}
s^6: & 1 & 3 & 1 & 4 \\
s^5: & 4 & 2 & 4 & 0 \\
s^4: & \frac{5}{2} = \frac{4.3-1.2}{4} & 0 = \frac{4.1-4.1}{4} & 4 = \frac{4.4-1.0}{4} & \\
s^3: & 2 = \frac{(5/2).2-4.0}{(5/2)} & -\frac{12}{5} = \frac{(5/2).4-4.4}{(5/2)} & 0 & \\
s^2: & 3 & 4 & & \\
s^1: & -\frac{76}{15} & 0 & & \\
s^0: & 4 & & &
\end{array}$$

- note the two sign changes in the first column – there are therefore 2 roots in RHP

15.1.1 Special case of the Routh criterion

- if the first element in a row is zero, but others are non-zero, replace 0 with ε > 0 and proceed as before. Then let ε $\rightarrow$ 0.

- for example:

$$A(s) = s^4 + 5s^3 + 7s^2 + 5s + 6 \tag{15.6}$$

- has the following Routh array:

$$\begin{array}{lll} s^4: & 1 & 7 \quad 6 \\ s^3: & 5 & 5 \\ s^2: & 6 & 6 \\ s^1: & \varepsilon & \\ s^0: & 6 & \end{array}$$

- since:
  1. if ε > 0, there are 0 sign changes (all stable roots)
  2. if ε < 0, there are 2 sign changes (2 unstable roots).

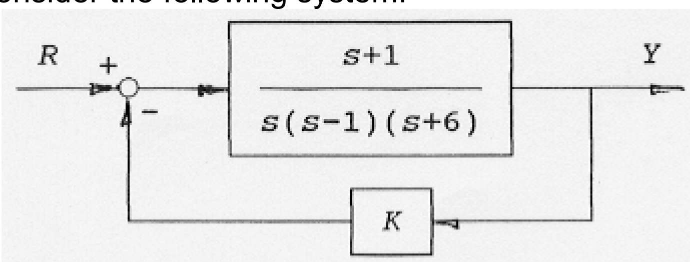
- thus, ε = 0 indicates 2 imaginary roots (zero real part) ie., two roots lie on the imaginary axis.

- the actual roots are $-3, -2, 0 \pm i$, as predicted

15.2 Conditions for stability

- the Routh stability criterion may also be used to find the conditions for stability, in terms of variable parameters.

- for example, consider the following system:



- the characteristic equation for this closed-loop system is

$$A(s) = 1 + \frac{K(s+1)}{s(s-1)(s+6)} = 0 \tag{15.7}$$

- that is,

$$s^3 + 5s^2 + (K-6)s + K = 0 \tag{15.8}$$

- in order not to have any negative coefficients, a necessary (but not sufficient) condition for stability is clearly that $K > 6$

- the Routh array is

$$\begin{array}{lll} s^3: & 1 & (K-6) \\ s^2: & 5 & K \\ s^1: & \dfrac{4K-30}{5} & \\ s^0: & K & \end{array}$$

- hence for stability, we also require that
  1. $K > 0$
  2. $4K - 30 > 0$

- thus, the conditions that $K > 6$ and $K > 0$ are satisfied, and the system will be stable, if $K > 7.5$, which is both a necessary and sufficient condition for stability.