Lecture 12: Bode plots

- Bode plots provide a standard format for presenting frequency response data. They consist of the variation of the amplitude ratio $\log_{10}\left[A(\omega)\right]$ and the relative phase $\varphi(\omega)$ versus the angular frequency $\log_{10}(\omega)$

- as discussed in the previous lecture, Bode plots represent the *steady-state response* to *sinusoidal* excitation only. However, information about the transient response or the response to non-sinusoidal excitation can often be inferred from the Bode plot, although these inferences are not always valid.

- Bode plots can be sketched rapidly using asymptotic approximations, and the frequency responses for higher order systems can be constructed by adding the curves for elementary factors of the transfer function.

12.1 General amplitude and phase relationships

- suppose that a transfer function $G(s)$ can be written as the product of a number of several other transfer functions:

$$G(s) = G_1(s)G_2(s)G_3(s)\dots \tag{12.1}$$

- the complex frequency response function is then obtained by substituting $s = i\omega$:

$$\begin{aligned} G(i\omega) &= G_1(i\omega)G_2(i\omega)G_3(i\omega)\dots \\ &= \left|G_1(i\omega)\right|e^{i\varphi_1(\omega)}\left|G_2(i\omega)\right|e^{i\varphi_2(\omega)}\left|G_3(i\omega)\right|e^{i\varphi_3(\omega)}\dots \\ &= G_1G_2G_3e^{i\left[\varphi_1(\omega)+\varphi_2(\omega)+\varphi_3(\omega)\right]}\dots \end{aligned} \tag{12.2}$$

- where $G_i = \left|G_i(i\omega)\right|$

- thus

$$\left|G(i\omega)\right|e^{i\varphi(\omega)} = G_1G_2G_3e^{i\left[\varphi_1(\omega)+\varphi_2(\omega)+\varphi_3(\omega)\right]}\dots \tag{12.3}$$

- so that

$$\begin{aligned} A(\omega) &= G_1G_2G_3\dots \\ \Rightarrow Lm \equiv \log_{10}\left[A(\omega)\right] &= \log_{10}\left[G_1\right] + \log_{10}\left[G_2\right] + \log_{10}\left[G_3\right]\dots \end{aligned} \tag{12.4}$$

- and

$$\varphi(\omega) = \mathrm{ph}\left[G(i\omega)\right] = \varphi_1(\omega) + \varphi_2(\omega) + \varphi_3(\omega) + \dots \tag{12.5}$$

- thus, on a Bode plot of the <u>log magnitude</u> Lm *vs* $\log_{10}(\omega)$ and the (relative) phase $\varphi(\omega)$ vs. $\log_{10}(\omega)$, the contributions of the individual factors are simply <u>added</u>.

- assume that a transfer function can be factored into the following form:

$$G(s) = \frac{Ke^{-\tau s} \prod_{i=1}^{p} (T_i s + 1) \prod_{k=1}^{(m-p)/2} \left[(s/\omega_k)^2 + 2\xi_k (s/\omega_k) + 1 \right]}{s^N \prod_{l=1}^{q} (T_l s + 1) \prod_{r=1}^{(n-q)/2} \left[(s/\omega_r)^2 + 2\xi_r (s/\omega_r) + 1 \right]} \tag{12.6}$$

- we shall refer to this form as the Bode form, and the gain K as the Bode gain.

- thus, the elementary factors are:

  1. Bode gain: K

  2. free s: if s^N occurs in the denominator, $G(s)$ has N poles at the origin (multiple integration); s^N in the numerator (N zeros at s=0) implies multiple differentiation.

  3. 1st order factor: a factor $(Ts+1)$ is referred to as a 1st-order 'lead' and the value $s = -1/T$ is a 'zero' of the transfer function $G(s)$. Correspondingly, a factor $(Ts+1)^{-1}$ is a 1st-order 'lag' and the value $s = -1/T$ is a 'pole'.

  4. 2nd order (quadratic) factor: a factor $\left[(s/\omega)^2 + 2\xi(s/\omega) + 1\right]$ is a second order lead with zeros at $s = -\xi\omega \pm \omega\sqrt{1-\xi^2}$. A factor $\left[(s/\omega)^2 + 2\xi(s/\omega) + 1\right]^{-1}$ is a second order lag with two poles at $s = -\xi\omega \pm \omega\sqrt{1-\xi^2}$.

  5. time delay (transport lag): $e^{-\tau s}$, since one of the properties of the Laplace transform is that if $\mathcal{L}\left[f(t)\right] = F(s)$ then $\mathcal{L}\left[f(t-\tau)\right] = e^{-\tau s}F(s)$

- consider the contribution that each of these factors makes to a Bode plot

12.2 Bode gain K

- transfer function

$$G(s) = K \tag{12.7}$$

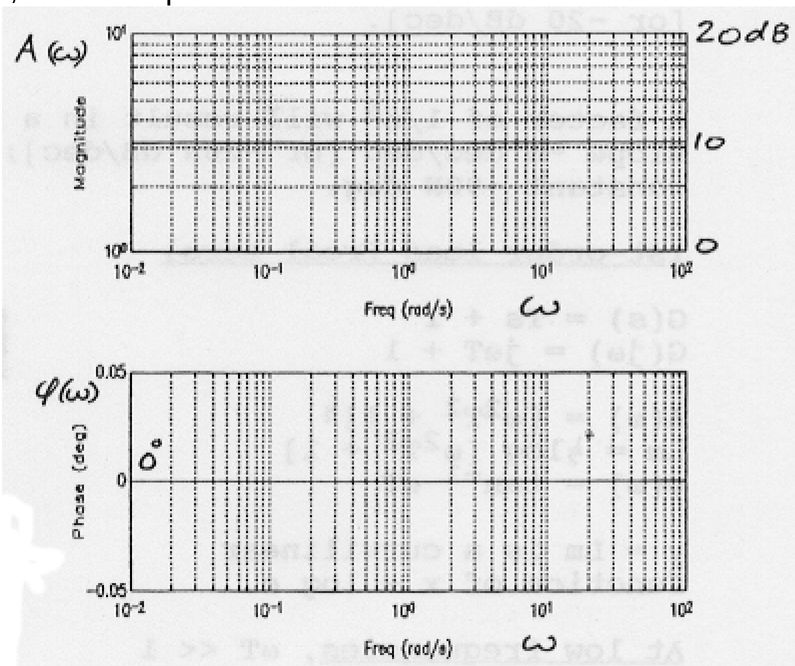
- thus

$$\begin{aligned} G(i\omega) &= K \\ \Rightarrow A(\omega) &= K \end{aligned} \tag{12.8}$$

- and

$$\varphi(\omega) = \begin{cases} 0 & if \;\; K > 0 \\ \pm\pi & if \;\; K < 0 \end{cases} \tag{12.9}$$

- if $K = 3.162$, the Bode plot is then



- note that it is usual to label the logarithmic axes with the actual amplitude ratio or frequency values, rather than their logarithms.

- it is also quite common for the amplitude ratio to be expressed in terms of <u>decibels</u>:

$$Lm\ dB = 20\log_{10}\left|G(i\omega)\right| = 20\ Lm \tag{12.10}$$

  - thus, in this example: $A(\omega) = 3.162 \Rightarrow Lm = 0.5 \Rightarrow Lm\ dB = 10dB$

12.3 Zero at origin

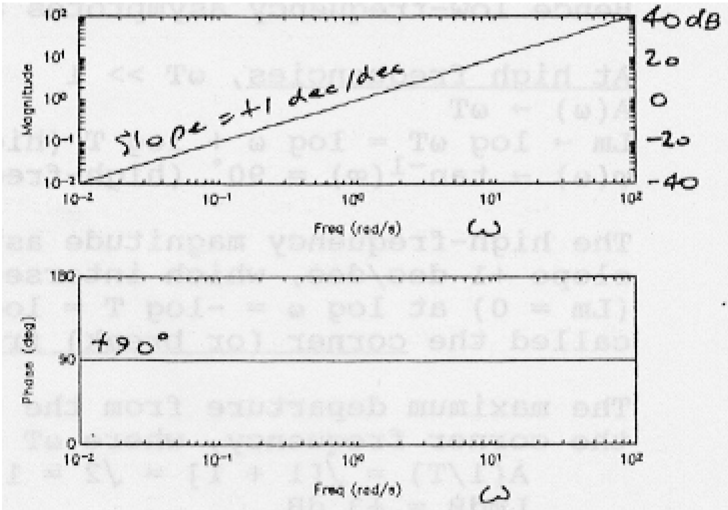
- transfer function

$$\begin{aligned} G(s) &= s \\ \Rightarrow G(i\omega) &= i\omega = \omega e^{i\pi/2} \end{aligned} \tag{12.11}$$

- thus

$$\begin{aligned} A(\omega) &= \omega \\ \&\ \varphi(\omega) &= \frac{\pi}{2} rads = +90^0 \end{aligned} \tag{12.12}$$

- since $A(\omega) = \omega \Rightarrow Lm = \log_{10}(\omega)$, giving a straight line in logarithmic axes with a gradient of $+1\ decade/decade = +20\ dB/decade$. The Bode plot is



- a factor of s^N will result in a straight-line magnitude plot of slope $+N \ dec/dec = +20N \ dB/dec$; the phase *lead* is constant at $+90N^0$

12.4 Pole at origin

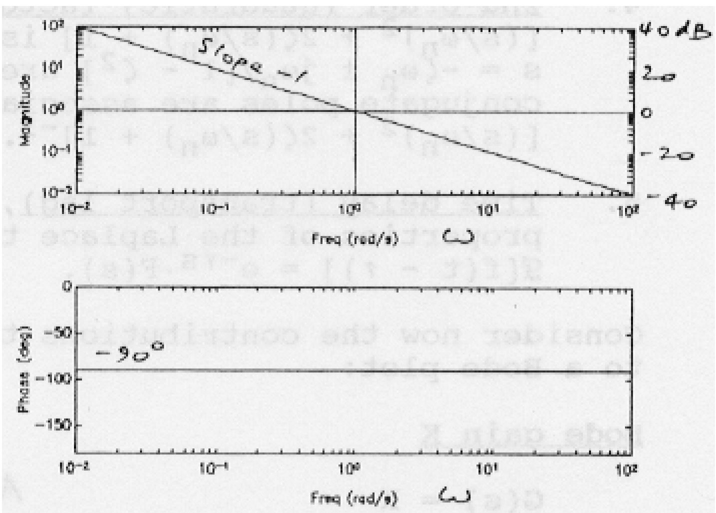
- transfer function

$$G(s) = \frac{1}{s}$$
$$\Rightarrow G(i\omega) = \frac{-i}{\omega} = \frac{1}{\omega} e^{-i\pi/2} \tag{12.13}$$

- thus

$$A(\omega) = \frac{1}{\omega}$$
$$\& \ \varphi(\omega) = -\frac{\pi}{2} rads = -90^0 \tag{12.14}$$

- since $A(\omega) = 1/\omega \Rightarrow Lm = -\log_{10}(\omega)$, giving a straight line in logarithmic axes with a gradient of $-1 \ dec/dec$:



- a factor of s^{-N} will result in a straight-line magnitude plot of slope $-N\ dec/dec = -20N\ dB/dec$; the phase *lag* is constant at $-90N^0$

12.5 1st order lead (real zero)

- transfer function

$$\begin{aligned} G(s) &= Ts+1 \\ \Rightarrow G(i\omega) &= iT\omega+1 \end{aligned} \tag{12.15}$$

- thus

$$\begin{aligned} A(\omega) &= \sqrt{(T\omega)^2+1} \\ \&\ \varphi(\omega) &= \operatorname{atan}(T\omega) \end{aligned} \tag{12.16}$$

- at *low* frequencies ($\omega \ll 1/T$), as $\omega \to 0$

$$\begin{aligned} A(\omega) &\to 1 \\ \&\ \varphi(\omega) &\to 0^0 \end{aligned} \tag{12.17}$$

- hence the low frequency asymptotes are $A(\omega)=1\ \&\ \varphi(\omega)=0^0$

- at *high* frequencies ($\omega \gg 1/T$), as $\omega \to \infty$

$$\begin{aligned} A(\omega) &\to T\omega \\ \&\ \varphi(\omega) &\to +90^0 \end{aligned} \tag{12.18}$$

- hence the high frequency asymptotes are $A(\omega)=T\omega\ \&\ \varphi(\omega)=+90^0$

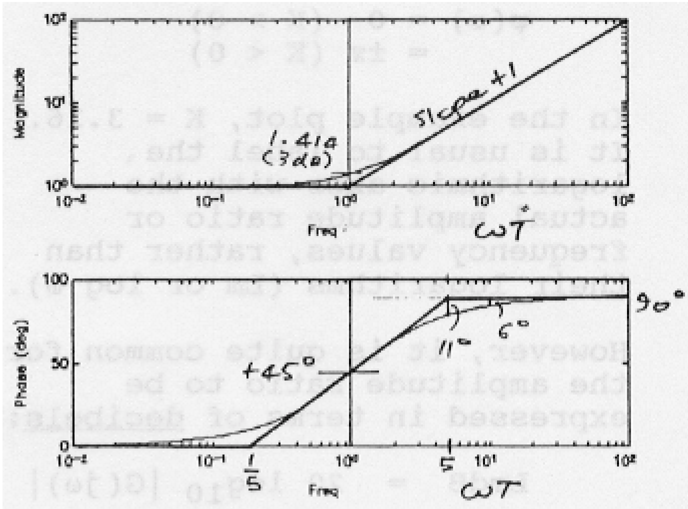
- the high frequency magnitude asymptote is a straight line of slope $+1\ dec/dec$, which intersects the low-frequency asymptote $(Lm=0)$ at $\log(\omega) = -\log(T) \Rightarrow \omega = 1/T$. The frequency $\omega = 1/T$ is called the <u>corner (or break) frequency</u>.

- the maximum departure from the two magnitude asymptotes occurs at the corner frequency:

$$\begin{aligned} A(1/T) &= \sqrt{1+1} = 1.414 = +3dB \\ \&\ \varphi(1/T) &= \operatorname{atan}(1) = +45^0 \end{aligned} \tag{12.19}$$

- note that the phase at the corner frequency is mid-way between those of the low and high frequency asymptotes. We can therefore sketch another phase asymptote along a line of slope $+45^0\ /dec$ through the corner frequency phase of $+45^0$, although it is emphasised that, unlike the high and low frequency asymptotes, this asymptote is only approximate

- the Bode plot is then



12.6 1st order lag (real pole)

- transfer function

$$G(s) = \frac{1}{Ts+1}$$
$$\Rightarrow G(i\omega) = \frac{1}{iT\omega+1} \tag{12.20}$$

- thus

$$A(\omega) = \left[(T\omega)^2 + 1\right]^{-1/2}$$
$$\& \ \varphi(\omega) = \operatorname{atan}(-T\omega) \tag{12.21}$$

- at *low* frequencies ($\omega \ll 1/T$), as $\omega \to 0$

$$A(\omega) \to 1$$
$$\& \ \varphi(\omega) \to 0^0 \tag{12.22}$$

- hence the low frequency asymptotes are again $A(\omega) = 1 \& \ \varphi(\omega) = 0^0$

- at *high* frequencies ($\omega \gg 1/T$), as $\omega \to \infty$

$$A(\omega) \to \frac{1}{T\omega}$$
$$\& \ \varphi(\omega) \to -90^0 \tag{12.23}$$

- hence the high frequency asymptotes are $A(\omega) = 1/T\omega \ \& \ \varphi(\omega) = -90^0$

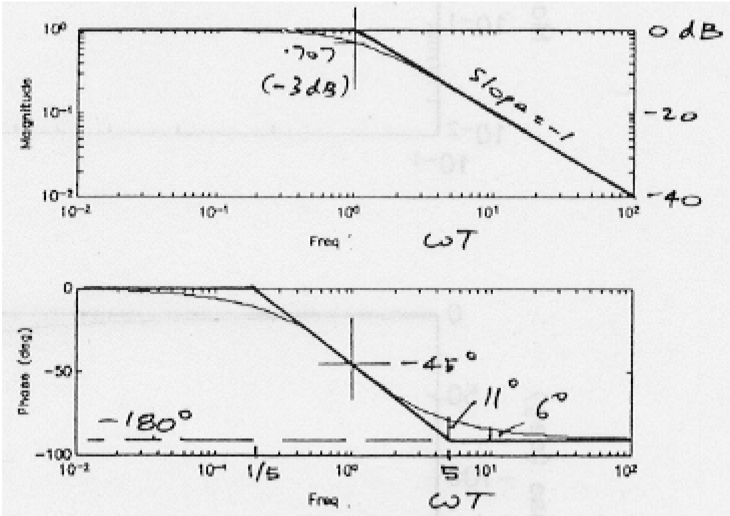
- the high frequency magnitude asymptote is a straight line of slope $-1 \ dec/dec$, which once again intersects the low-frequency asymptote ($Lm = 0$) at the corner frequency $\log(\omega) = -\log(T) \Rightarrow \omega = 1/T$.

- at the corner frequency:

$$
\begin{aligned}
A(1/T) &= 1/\sqrt{1+1} = 0.7071 = -3\ dB \\
\&\ \varphi(1/T) &= \operatorname{atan}(1) = -45^0
\end{aligned}
\tag{12.24}
$$

- note that we now sketch the corner frequency phase asymptote along a line of slope $-45^0\ /dec$ through the corner frequency phase of -45^0.

- the Bode plot is



12.7 2nd order lag (quadratic pole)

- transfer function

$$
\begin{aligned}
G(s) &= \frac{1}{(s/\omega_n)^2 + 2\xi(s/\omega_n) + 1} \\
\Rightarrow G(i\omega) &= \frac{1}{1-(\omega/\omega_n)^2 + i2\xi\omega/\omega_n}
\end{aligned}
\tag{12.25}
$$

- thus

$$
\begin{aligned}
A(\omega) &= \frac{1}{\sqrt{\left[1-(\omega/\omega_n)^2\right]^2 + 4\xi^2(\omega/\omega_n)^2}} \\
\&\ \varphi(\omega) &= \operatorname{atan}\left(\frac{-2\xi(\omega/\omega_n)}{\left[1-(\omega/\omega_n)^2\right]}\right)
\end{aligned}
\tag{12.26}
$$

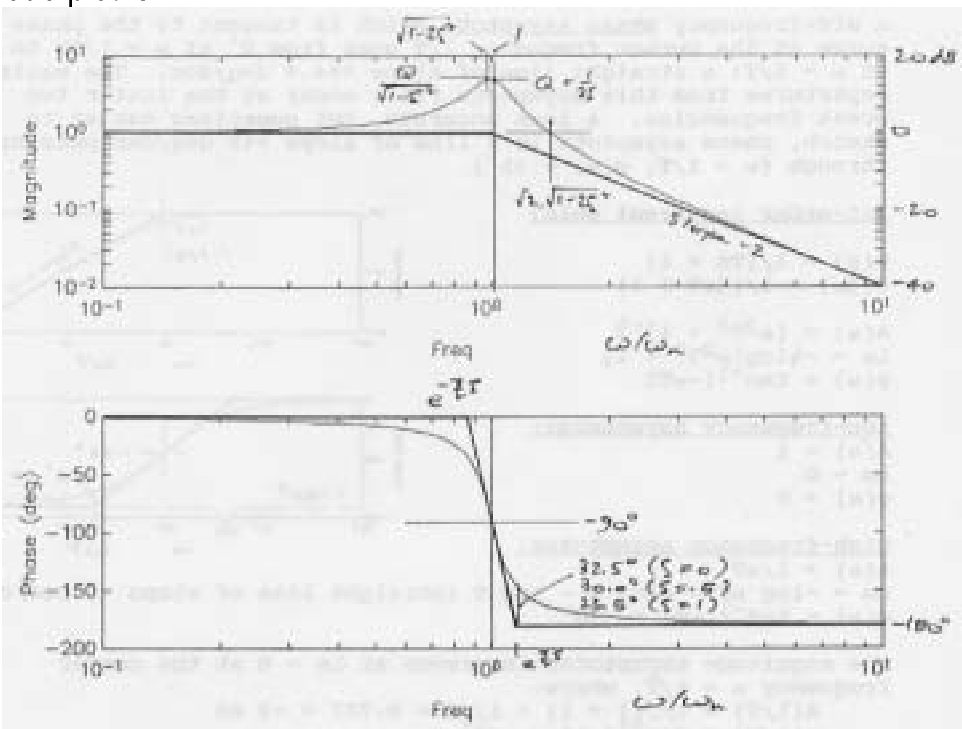
- at *low* frequencies ($\omega \ll \omega_n$), as $\omega \to 0$

$$
\begin{aligned}
A(\omega) &\to 1 \\
\&\ \varphi(\omega) &\to 0^0
\end{aligned}
\tag{12.27}
$$

- at *high* frequencies ($\omega \gg \omega_n$), as $\omega \to \infty$

$$A(\omega) \to \frac{1}{(\omega/\omega_n)^2}$$
$$\& \ \varphi(\omega) \to -180^0 \tag{12.28}$$

- the Bode plot is



- the high-frequency magnitude asymptote is a straight line of slope $-2\ dec/dec = -40\ dB/dec$, intersecting the low-frequency asymptote at the corner frequency $\omega = \omega_n$. At the corner frequency:

$$G(i\omega_n) = \frac{1}{i2\xi}$$
$$\Rightarrow A(\omega_n) = \frac{1}{2\xi} \tag{12.29}$$
$$\& \ \varphi(\omega_n) = -90^0$$

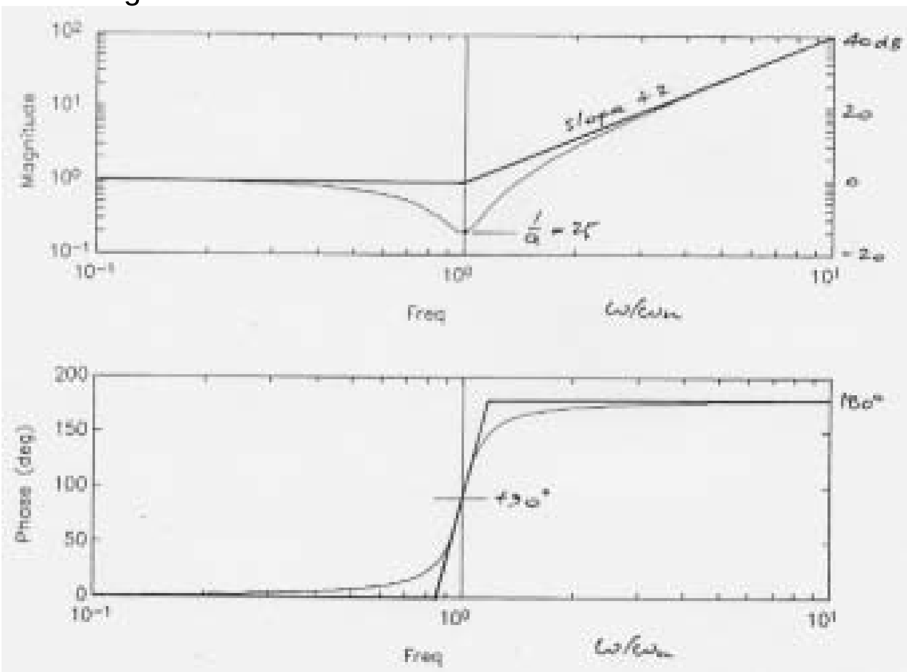
- note that the term 'quality factor' $Q \equiv A(\omega_n) = \dfrac{1}{2\xi}$ is often used to determine the amplitude of the response when the system is excited at the undamped natural frequency ω_n

12.8 2nd order lead (quadratic zero)

- transfer function

$$G(s) = (s/\omega_n)^2 + 2\xi(s/\omega_n) + 1 \tag{12.30}$$

- the Bode plot for this numerator factor can be sketched using the same rules as for a 2nd order lag, by simply inverting the amplitude ratio, and changing the sign of the phase angle:



12.9 Time delay

- transfer function

$$\begin{aligned} G(s) &= e^{-\tau s} \\ \Rightarrow G(i\omega) &= e^{-i\tau\omega} \end{aligned} \tag{12.31}$$

- thus

$$\begin{aligned} A(\omega) &= 1 \\ \& \ \varphi(\omega) &= -\tau\omega \end{aligned} \tag{12.32}$$

- thus a pure time delay (or 'transport lag') results in a phase shift which increases linearly with frequency (curved on logarithmic axes):