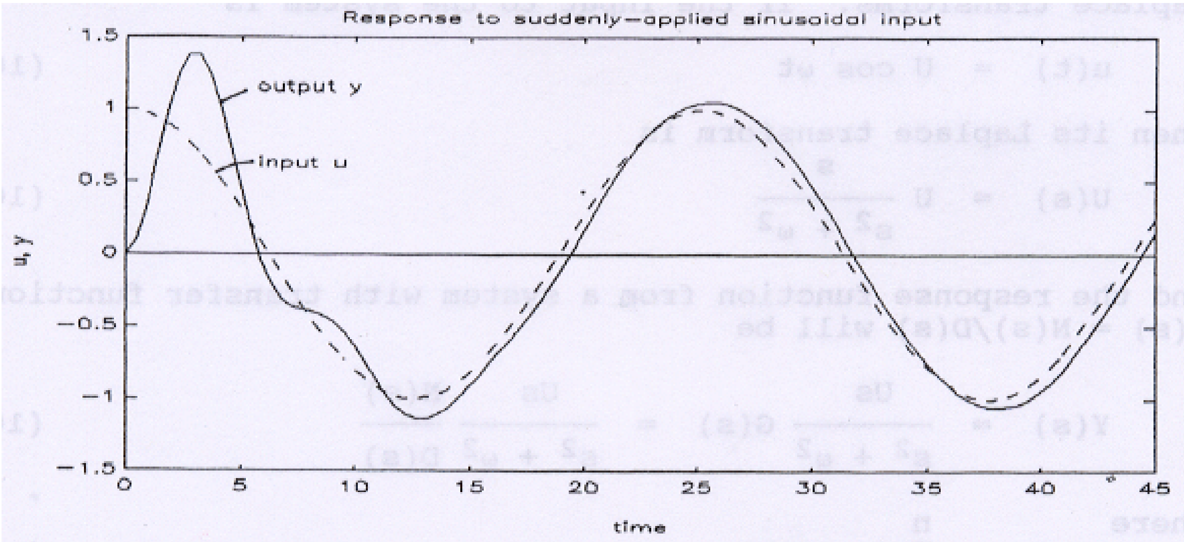
Lecture 11: Frequency response – introduction to Bode plots

- when a time-invariant, stable linear system is excited by a sinusoidal input of a given frequency, its 'steady-state' response (ie. after 'transient' motions have decayed) can be shown to also be sinusoidal at the same frequency.

- note, however, that the amplitude (and, typically, the units) of the output will be different from the input, and the output sinusoid will generally be phase-shifted with respect to the input.

- these characteristics can be seen if we consider the response of a simple second-order system ($\omega_n = 1,\ \xi = 0.2$) to a cosine wave of angular frequency $\omega = 0.25$ applied at $t = 0$:



- the term 'frequency response' means the variation of the output amplitude and phase with frequency. We shall see that a number of control design methods are based on a frequency response representation of the system dynamics.

11.1 Phasor representation of the transfer function

- we saw in Lecture 4 that the amplitude and phase of a sinusoidal signal $y(t) = Y\cos(\omega t + \varphi)$ can be represented by a complex phasor $\mathbb{Y}$:

$$\mathbb{Y} = Ye^{i\varphi}$$
$$\Rightarrow y(t) = \mathrm{Re}\left\{\mathbb{Y}e^{st}\right\} \qquad (11.1)$$

- where $s = i\omega$

- consider now the sinusoidal input to a system described by a transfer function $G(s) = Y(s)/U(s)$:

$$u(t) = \mathbb{U}e^{st} \qquad (11.2)$$

- where $s = i\omega$ & $\mathbb{U} = Ue^{i0} = U$ (note that the phase of the input signal is defined as the reference, and so has zero phase)

- the steady-state sinusoidal response will be then be given by

$$y(t) = \mathbb{Y}e^{st} = G(s)\mathbb{U}e^{st} \tag{11.3}$$

- where $\mathbb{Y} = Ye^{i\varphi}$ & $s = i\omega$

- the input and output phasors are therefore related by

$$\frac{\mathbb{Y}}{\mathbb{U}} = \frac{Y}{U}e^{i\varphi} = G(s) \tag{11.4}$$

- with $s = i\omega$. When the transfer function $G(s)$ is evaluated for $s = i\omega$, it is referred to as the complex frequency response function $G(i\omega)$.

- hence, the amplitude ratio between the input and output *at a given frequency* is

$$A(\omega) = \frac{Y}{U} = \left|G(i\omega)\right| \tag{11.5}$$

- and the relative phase is

$$\varphi(\omega) = \mathrm{ph}\left[G(i\omega)\right] \tag{11.6}$$

11.2 Relationship between phasor and s-plane representations

- the amplitude ratio (11.5) and relative phase relationships (11.6) can be established more formally using Laplace transforms. Specifically, the previously assumed sinusoidal response to sinusoidal forcing can be shown mathematically

- the input to the system

$$u(t) = U\cos(\omega t) \tag{11.7}$$

- has the Laplace transform:

$$U(s) = \frac{Us}{s^2 + \omega^2} \tag{11.8}$$

- and the response function from a system with the general transfer function $G(s) = N(s)/D(s)$ will be

$$Y(s) = \frac{Us}{s^2 + \omega^2}\frac{N(s)}{D(s)} \tag{11.9}$$

- where

$$N(s) = \prod_{i=1}^{m}(s + z_i) \;\&\; D(s) = \prod_{j=1}^{n}(s + p_j) \tag{11.10}$$

- assuming that $m \le n$, a partial fraction expansion of (11.9) gives

$$Y(s) = \frac{a}{s - i\omega} + \frac{a^*}{s + i\omega} + \sum_{i=1}^{n} \frac{a_i}{s + p_i} \tag{11.11}$$

- with the time response determined from the inverse transform:

$$y(t) = ae^{i\omega t} + a^* e^{-i\omega t} + \sum_{i=1}^{n} a_i e^{-p_i t} \tag{11.12}$$

- for a stable system, the poles $s = -p_i$ will have negative real parts, and the transients represented by the summation in (11.12) will therefore decay exponentially to zero, leaving the 'steady state' motion:

$$y_{ss}(t) = ae^{i\omega t} + a^* e^{-i\omega t} \tag{11.13}$$

- with the steady state response function

$$Y_{ss}(s) = \frac{a}{s - i\omega} + \frac{a^*}{s + i\omega} \tag{11.14}$$

- combining equations (11.9) and (11.14) at $s = i\omega$

$$\left[\frac{a}{s - i\omega} + \frac{a^*}{s + i\omega}\right]_{s=i\omega} = \left[\frac{Us}{s^2 + \omega^2} G(s)\right]_{s=i\omega} \tag{11.15}$$

- both the first term on the LHS and the RHS of (11.15) are undefined, whilst the second term on the LHS is finite. The second term can then be ignored, giving

$$\begin{aligned} a &= \left[\frac{Us(s - i\omega)}{s^2 + \omega^2} G(s)\right]_{s=i\omega} \\ &= \left[\frac{Us}{s + i\omega} G(s)\right]_{s=i\omega} \\ &= \frac{1}{2} UG(i\omega) \end{aligned} \tag{11.16}$$

- defining

$$\begin{aligned} a &= \frac{1}{2} U \left|G(i\omega)\right| e^{i\varphi(\omega)} \\ a^* &= \frac{1}{2} U \left|G(i\omega)\right| e^{-i\varphi(\omega)} \end{aligned} \tag{11.17}$$

- where $\varphi(\omega) = \mathrm{ph}\left[G(i\omega)\right]$. Using (11.13), we then get

$$\begin{aligned} y_{ss}(t) &= \frac{1}{2} U \left|G(i\omega)\right| \left[e^{i(\omega t + \varphi)} + e^{-i(\omega t + \varphi)}\right] \\ &= U \left|G(i\omega)\right| \cos\left[\omega t + \varphi(\omega)\right] \\ &= Y \cos\left[\omega t + \varphi(\omega)\right] \end{aligned} \tag{11.18}$$

- thus, the steady-state response to a sinusoidal input is also sinusoidal at the same frequency.

- the <u>amplitude ratio</u> $A(\omega)$ is given by

$$A(\omega) = \frac{Y}{U} = \left|G(i\omega)\right| \tag{11.19}$$

- and the <u>relative phase</u> $\varphi(\omega)$ is

$$\varphi(\omega) = \mathrm{ph}\left[G(i\omega)\right] \tag{11.20}$$

- which are the same as those given in equations (11.5) and (11.6).

11.3 Example: the simple 2nd order system

- consider the 2nd order system with transfer function

$$G(s) = \frac{\omega_n^{\;2}}{s^2 + 2\xi\omega_n s + \omega_n^{\;2}} \tag{11.21}$$

- the complex frequency response function $G(i\omega)$ is $G(s)$ evaluated along the imaginary ($i\omega$) axis: