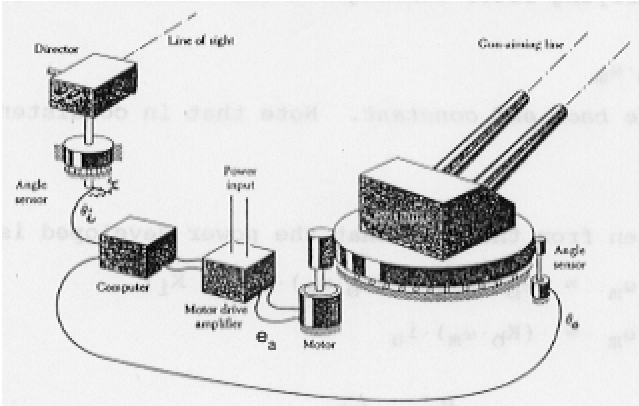
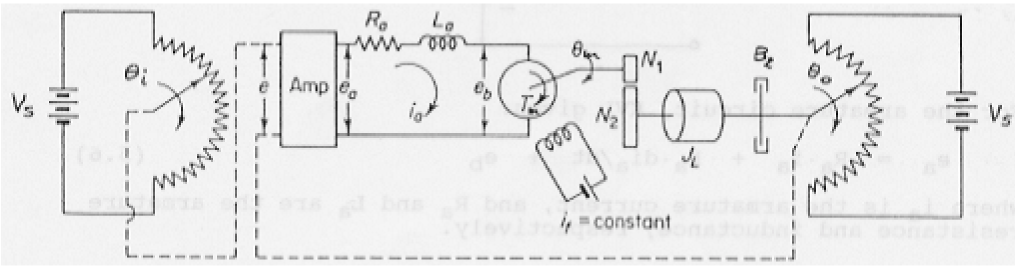
Lecture 10: Time response – effect of extra poles and zeros

10.1 The electro-mechanical servo

- consider the following electro-mechanical servo, where the task of the servo is to make the gun output angle θ_o track the detector input angle θ_i:



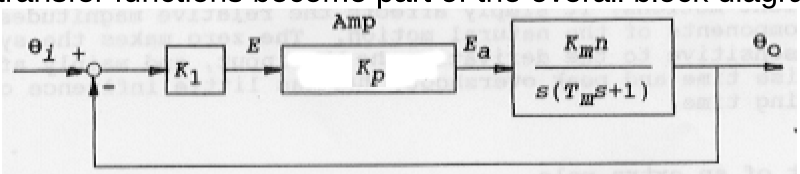
- the circuit diagram of this system is



- this circuit can be broken into the following transfer functions:

error detector	$E = K_1\left(\theta_i - \theta_0\right)$
controller amplifier	$E_a = K_p E$
motor (armature controlled)	$G_m = \dfrac{\theta_m}{E_a} = \dfrac{K_m}{s\left(T_m s + 1\right)}$
gears	$\theta_0 = n\theta_m$

- and these transfer functions become part of the overall block diagram:



- using results to be presented in later lectures, this block diagram represents a unity gain feedback system with a 2nd order, closed-loop transfer function:

$$\frac{\theta_o(s)}{\theta_i(s)} = \frac{K_1 K_p K_m n}{T_m s^2 + s + K_1 K_p K_m n} = \frac{\omega_n^{\;2}}{s^2 + 2\xi\omega_n s + \omega_n^{\;2}} \qquad (10.1)$$

- thus, given values of the constants $K_l,\ K_p,\ K_m,\ n$, the behaviour of the system is fixed since:

$$\omega_n^{\ 2} = \frac{K_1 K_p K_m n}{T_m}$$
$$\& \ \xi = \frac{1}{2\omega_n T_m} = \frac{1}{2T_m}\sqrt{\frac{T_m}{K_1 K_p K_m n}} \tag{10.2}$$

10.2 Effect of a zero in transfer function

- we may wish to improve the system behaviour in many cases. A common strategy for increasing the responsiveness of this system, and at the same time increasing the closed-loop damping, is to make the control signal e_a sensitive to the *rate of change* of the error, as well as its instantaneous value:

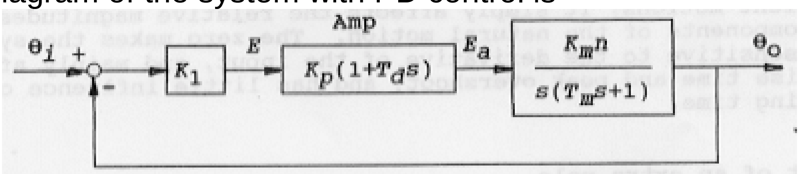
$$e_a(t) = K_p e(t) + K_d \dot{e}(t) \tag{10.3}$$

- where K_p is the *proportional gain* and K_d is the *derivative gain*. This control strategy is called proportional + derivative (PD) control, and has the controller transfer function

$$\frac{E_a(s)}{E(s)} = K_p + K_d s = K_p\left(1 + T_d s\right) \tag{10.4}$$

- where $T_d = K_d / K_p$ is the *derivative time*.

- the block diagram of the system with PD control is



- with closed-loop transfer function

$$\frac{\theta_o(s)}{\theta_i(s)} = \frac{K_1 K_p K_m n\left(1 + T_d s\right)}{T_m s^2 + \left(1 + K_1 K_p K_m n T_d\right)s + K_1 K_p K_m n} \tag{10.5}$$

- recalling that the roots of the characteristic equation determine the nature of the transient response:

$$s^2 + 2\xi\omega_n s + \omega_n^{\ 2} = 0 \tag{10.6}$$

- where:

$$\omega_n^{\ 2} = \frac{K_1 K_p K_m n}{T_m}$$
$$\& \ \xi = \frac{\left(1 + K_1 K_p K_m n T_d\right)}{2\omega_n} = \frac{\left(1 + K_1 K_p K_m n T_d\right)}{2T_m}\sqrt{\frac{T_m}{K_1 K_p K_m n}} \tag{10.7}$$

- comparing (10.5) with the transfer function (10.1) for proportional control only (ie. T_d = 0), we see that a zero at s = -1/T_d has been introduced. As (10.7) and (10.2) then show, the damping has been increased.

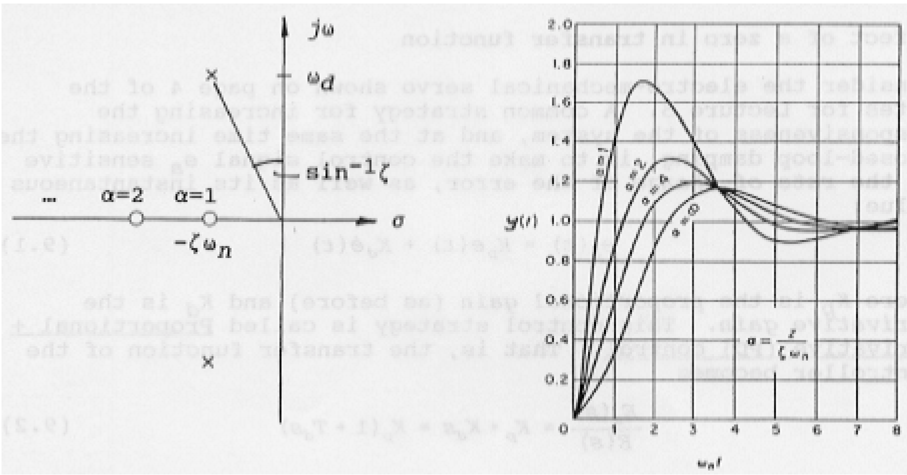
- generalising from (10.5), we can write the transfer function for a second-order system with one zero as:

$$\frac{Y(s)}{U(s)}=\frac{\omega_n^{\,2}}{\alpha}\frac{s+\alpha}{\left(s^2+2\xi\omega_n s+\omega_n^{\,2}\right)} \qquad (10.8)$$

- the partial fraction expansion of the step response function (ie. $U(s)=1/s$) is

$$Y(s)=\frac{1}{s}-\frac{s}{\left(s^2+2\xi\omega_n s+\omega_n^{\,2}\right)}+\left(\frac{1}{\alpha}-\frac{2\xi}{\omega_n}\right)\frac{\omega_n^{\,2}}{\left(s^2+2\xi\omega_n s+\omega_n^{\,2}\right)} \qquad (10.9)$$

- from which it is apparent that when α is small, (i.e. when the zero is close to the origin in the s-plane) the last term on the right side of (10.9) will dominate the step response. This effect is illustrated below for several locations of the zero relative to the complex poles, for a system with $\xi=0.5$:

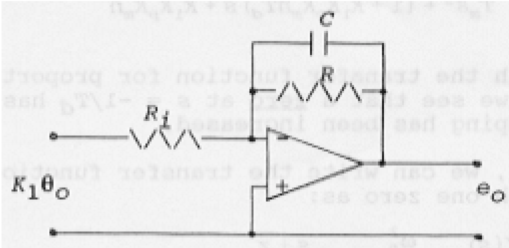


- where the case $\alpha=\infty$ corresponds to $T_d=1/\alpha=0$

- note that the zero has no effect on the character of the transient motions; it simply affects the relative magnitudes of the components of the natural motion. The zero makes the system more sensitive to the derivative of the input, and mainly affects the rise time and peak overshoot.

10.3 Effect of an extra pole

- consider again the electro-mechanical servo system. Suppose that the voltage from the output angle potentiometer is low-pass filtered:



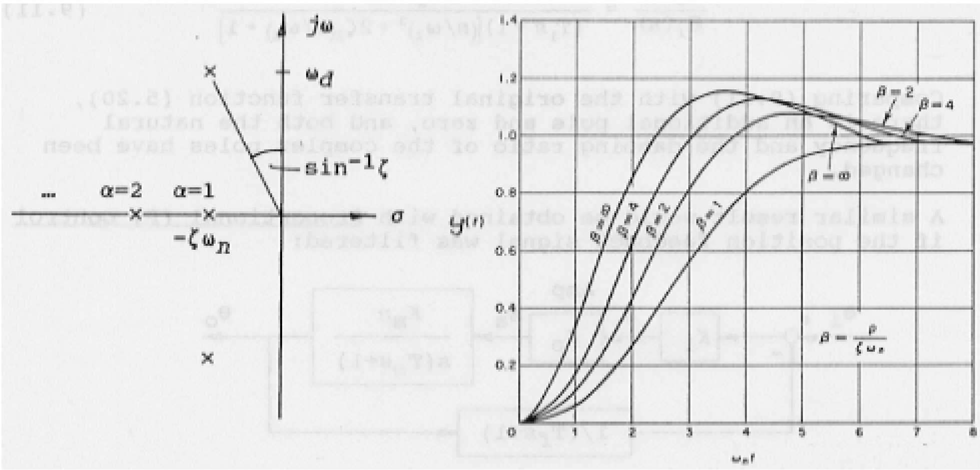
- the transfer function for this filter is

$$\frac{E_0(s)}{K_1\theta_0(s)} = -\frac{R/R_i}{RCs+1} \tag{10.10}$$

- hence, using equation (10.1), the transfer function from the input potentiometer voltage $e_i = K_1\theta_i$ to the filtered output voltage e_o is of the form

$$\frac{E_o(s)}{E_i(s)} = \frac{E_o}{\theta_o}\frac{\theta_i}{E_i}\frac{\theta_o}{\theta_i} \propto \frac{\omega_n^{\,2}\beta}{(s+\beta)\left(s^2+2\xi\omega_n s+\omega_n^{\,2}\right)} \tag{10.11}$$

- where $\beta = 1/RC$. The effect on the step response of the additional <u>pole</u> at $s=-\beta$ is to reduce the overshoot and increase the rise time, as illustrated below for a system with $\xi = 0.5$:



- thus, in general, the effect of additional real poles or zeros will be most pronounced if they are ‘close’ to the imaginary axis.

10.4 Step response of a general system

- consider the general transfer function

$$G(s) = \frac{K\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{q}(s+p_j)\prod_{k=1}^{r}\left(s^2+2\xi_k\omega_{nk}s+\omega_{nk}^{\,2}\right)} \tag{10.12}$$

- which has *m* zeros (each of which may be real or complex), *q* real poles, and *r* pairs of complex conjugate poles.

- the order of the numerator and denominator is therefore m and $n = q + 2r$ respectively

- the general response function to a unit step $U(s) = 1/s$ is then

$$\begin{aligned} Y(s) &= G(s)/s \\ &= \frac{K\prod_{i=1}^{m}(s+z_i)}{s\prod_{j=1}^{q}(s+p_j)\prod_{k=1}^{r}(s^2+2\xi_k\omega_{nk}s+\omega_{nk}{}^2)} \end{aligned} \tag{10.13}$$

- assuming distinct poles, partial fraction expansion of (10.13) gives

$$Y(s) = \frac{a}{s} + \sum_{j=1}^{q}\frac{a_j}{s+p_j} + \sum_{k=1}^{r}\frac{b_k(s+\xi_k\omega_{nk}) + c_k\omega_{nk}\sqrt{1-\xi_k{}^2}}{s^2+2\xi_k\omega_{nk}s+\omega_{nk}{}^2} \tag{10.14}$$

- inverse Laplace transforming (10.14) yields the unit step response

$$\begin{aligned} y(t) &= a + \sum_{j=1}^{q} a_j e^{-p_j t} + \sum_{k=1}^{r} b_k e^{-\xi_k\omega_{nk}t}\cos(\omega_{dk}t) + \sum_{k=1}^{r} c_k e^{-\xi_k\omega_{nk}t}\sin(\omega_{dk}t) \\ &= a + \sum_{j=1}^{q} a_j e^{-p_j t} + \sum_{k=1}^{r} d_k e^{-\xi_k\omega_{nk}t}\sin(\omega_{dk}t+\phi_k) \end{aligned} \tag{10.15}$$

- where $\omega_{dk} = \omega_{nk}\sqrt{1-\xi_k{}^2}$

- note that the first term on the RHS of (10.15) is due to the pole from the input function. The inverse Laplace transform of this term yields the forced response, or particular solution, which can be shown using the final value theorem:

$$\begin{aligned} y_f &= a \\ &= \lim_{s\to 0} sY(s) = G(0) = \frac{K\prod_{i=1}^{m} z_i}{\prod_{j=1}^{q} p_j \prod_{k=1}^{r}\omega_{nk}{}^2} \end{aligned} \tag{10.16}$$

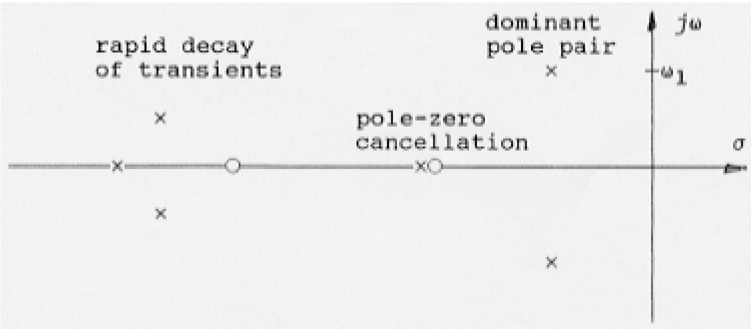
- the remaining terms in (10.15), which are due to the poles of the transfer function, give rise to a transient response which is the sum of a number of first and second order system responses. The magnitudes of the coefficients a_j, b_k ,c_k, d_k are influenced by the zeros of $G(s)$.

10.5 Dominant poles

- provided all the poles of $G(s)$ have negative real parts, the exponential terms in (10.15) will decay to zero. This decay will be most rapid for terms associated with poles that have large negative real parts (the so-called ‘fast poles’ discussed in lecture 8). Poles closer to the origin are likely to have a greater influence on the transient response.

- if a zero is close to a pole, the coefficient associated with that pole will be small and the corresponding transient term will be small; i.e., the zero effectively 'cancels' the pole.
  - this can be understood by considering the situation when a pole and zero occur at the exactly same place. The pole/zero pair will then cancel in equation (10.13), and will have be no effect whatsoever on the transient response (10.15).

- thus it <u>may</u> be that some poles close to the origin will 'dominate' the response:



- in this case, the response will be dominated by the complex conjugate poles nearest the origin, and the system will behave very like a simple second-order system with a damped natural frequency of ω_1. The well-known response characteristics for such systems can then be used to assess the system performance. It is quite common to *design* a control system to achieve this situation.

- utilising the notion of dominant poles is useful for preliminary design purposes. It is important, however, to check the validity of such simplifying assumptions by calculating the actual system response, based on the complete transfer function.