in Figure 2.1(*b*), where a mathematical function, called a transfer function, is inside each block, and block functions can easily be combined to yield Figure 2.1 (*a*) for ease of analysis and design. This convenience cannot be obtained with the differential equation.

◖ 2.2   Laplace Transform Review

A system represented by a differential equation is difficult to model as a block diagram. Thus, we now lay the groundwork for the Laplace transform, with which we can represent the input, output, and system as separate entities. Further, their interrelationship will be simply algebraic. Let us first define the Laplace transform and then show how it simplifies the representation of physical systems (*Nilsson, 1996*).

The Laplace transform is defined as

$$\mathscr{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}\, dt \qquad (2.1)$$

where $s = \sigma + j\omega$, a complex variable. Thus, knowing $f(t)$ and that the integral in Eq. (2.1) exists, we can find a function, $F(s)$, that is called the *Laplace transform* of $f(t)$.[1]

The notation for the lower limit means that even if $f(t)$ is discontinuous at $t = 0$, we can start the integration prior to the discontinuity as long as the integral converges. Thus, we can find the Laplace transform of impulse functions. This property has distinct advantages when applying the Laplace transform to the solution of differential equations where the initial conditions are discontinuous at $t = 0$. Using differential equations, we have to solve for the initial conditions after the discontinuity knowing the initial conditions before the discontinuity. Using the Laplace transform we need only know the initial conditions before the discontinuity. See *Kailath (1980)* for a more detailed discussion.

The inverse Laplace transform, which allows us to find $f(t)$ given $F(s)$, is

$$\mathscr{L}^{-1}[F(s)] = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st} ds = f(t)u(t) \qquad (2.2)$$

where

$$u(t) = 1 \quad t > 0$$
$$= 0 \quad t < 0$$

is the unit step function. Multiplication of $f(t)$ by $u(t)$ yields a time function that is zero for $t < 0$.

Using Eq. (2.1), it is possible to derive a table relating $f(t)$ to $F(s)$ for specific cases. Table 2.1 shows the results for a representative sample of functions. If we use the tables, we do not have to use Eq. (2.2), which requires complex integration, to find $f(t)$ given $F(s)$.

---

[1] The Laplace transform exists if the integral of Eq. (2.1) converges. The integral will converge if $\int_{0-}^{\infty} |f(t)|e^{-\sigma_1 t}\, dt < \infty$. If $|f(t)| < Me^{\sigma_2 t}, 0 < t < \infty$, the integral will converge if $\infty > \sigma_1 > \sigma_2$. We call σ_2 the *abscissa of convergence*, and it is the smallest value of σ, where $s = \sigma + j\omega$, for which the integral exists.

TABLE 2.1   Laplace transform table

Item no.	$f(t)$	$F(s)$
1.	$\delta(t)$	1
2.	$u(t)$	$\dfrac{1}{s}$
3.	$tu(t)$	$\dfrac{1}{s^2}$
4.	$t^n u(t)$	$\dfrac{n!}{s^{n+1}}$
5.	$e^{-at}u(t)$	$\dfrac{1}{s+a}$
6.	$\sin \omega t u(t)$	$\dfrac{\omega}{s^2+\omega^2}$
7.	$\cos \omega t u(t)$	$\dfrac{s}{s^2+\omega^2}$

In the following example we demonstrate the use of Eq. (2.1) to find the Laplace transform of a time function.

Example 2.1

Laplace Transform of a Time Function

PROBLEM: Find the Laplace transform of $f(t) = Ae^{-at}u(t)$.

SOLUTION: Since the time function does not contain an impulse function, we can replace the lower limit of Eq. (2.1) with 0. Hence,

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt = \int_0^\infty Ae^{-at}e^{-st}\,dt = A\int_0^\infty e^{-(s+a)t}\,dt$$

$$= -\frac{A}{s+a}e^{-(s+a)t}\bigg|_{t=0}^\infty = \frac{A}{s+a} \qquad (2.3)$$

In addition to the Laplace transform table, Table 2.1, we can use Laplace transform theorems, listed in Table 2.2, to assist in transforming between $f(t)$ and $F(s)$. In the next example, we demonstrate the use of the Laplace transform theorems shown in Table 2.2 to find $f(t)$ given $F(s)$.

Example 2.2

Inverse Laplace Transform

PROBLEM: Find the inverse Laplace transform of $F_1(s) = 1/(s+3)^2$.

SOLUTION: For this example we make use of the frequency shift theorem, Item 4 of Table 2.2, and the Laplace transform of $f(t) = tu(t)$, Item 3 of Table 2.1. If the inverse transform of $F(s) = 1/s^2$ is $tu(t)$, the inverse transform of $F(s+a) = 1/(s+a)^2$ is $e^{-at}tu(t)$. Hence, $f_1(t) = e^{-3t}tu(t)$.

TABLE 2.2 Laplace transform theorems

Item no.	Theorem	Name
1.	$\mathscr{L}[f(t)] = F(s) \ = \int_{0-}^{\infty} f(t)e^{-st}dt$	Definition
2.	$\mathscr{L}[kf(t)] \qquad = kF(s)$	Linearity theorem
3.	$\mathscr{L}[f_1(t) + f_2(t)] = F_1(s) + F_2(s)$	Linearity theorem
4.	$\mathscr{L}[e^{-at}f(t)] \quad = F(s+a)$	Frequency shift theorem
5.	$\mathscr{L}[f(t-T)] \quad = e^{-sT}F(s)$	Time shift theorem
6.	$\mathscr{L}[f(at)] \qquad = \frac{1}{a}F\left(\frac{s}{a}\right)$	Scaling theorem
7.	$\mathscr{L}\left[\dfrac{df}{dt}\right] \quad = sF(s) - f(0-)$	Differentiation theorem
8.	$\mathscr{L}\left[\dfrac{d^2f}{dt^2}\right] \quad = s^2F(s) - sf(0-) - f'(0-)$	Differentiation theorem
9.	$\mathscr{L}\left[\dfrac{d^nf}{dt^n}\right] \quad = s^nF(s) - \sum\limits_{k=1}^{n} s^{n-k}f^{k-1}(0-)$	Differentiation theorem
10.	$\mathscr{L}\left[\int_{0-}^{t} f(\tau)d\tau\right] = \dfrac{F(s)}{s}$	Integration theorem
11.	$f(\infty) \qquad = \lim\limits_{s \to 0} sF(s)$	Final value theorem[1]
12.	$f(0+) \qquad = \lim\limits_{s \to \infty} sF(s)$	Initial value theorem[2]

[1]For this theorem to yield correct finite results, all roots of the denominator of $F(s)$ must have negative real parts, and no more than one can be at the origin.

[2]For this theorem to be valid, $f(t)$ must be continuous or have a step discontinuity at $t = 0$ (that is, no impulses or their derivatives at $t = 0$).

Partial-Fraction Expansion

To find the inverse Laplace transform of a complicated function, we can convert the function to a sum of simpler terms for which we know the Laplace transform of each term. The result is called a *partial-fraction expansion*. If $F_1(s) = N(s)/D(s)$, where the order of $N(s)$ is less than the order of $D(s)$, then a partial-fraction expansion can be made. If the order of $N(s)$ is greater than or equal to the order of $D(s)$, then $N(s)$ must be divided by $D(s)$ successively until the result has a remainder whose numerator is of order less than its denominator. For example, if

$$F_1(s) = \frac{s^3 + 2s^2 + 6s + 7}{s^2 + s + 5} \tag{2.4}$$

we must perform the indicated division until we obtain a remainder whose numerator is of order less than its denominator. Hence,

$$F_1(s) = s + 1 + \frac{2}{s^2 + s + 5} \tag{2.5}$$

Taking the inverse Laplace transform, using Item 1 of Table 2.1, along with the differentiation theorem (Item 7) and the linearity theorem (Item 3 of Table 2.2), we obtain

$$f_1(t) = \frac{d\delta(t)}{dt} + \delta(t) + \mathscr{L}^{-1}\left[\frac{2}{s^2 + s + 5}\right] \tag{2.6}$$

Using partial-fraction expansion, we will be able to expand functions like $F(s) = 2/(s^2 + s + 5)$ into a sum of terms and then find the inverse Laplace transform for each term. We will now consider three cases and show for each case how an $F(s)$ can be expanded into partial fractions.

Case 1. Roots of the Denominator of *F(s)* Are Real and Distinct   An example of an $F(s)$ with real and distinct roots in the denominator is

$$F(s) = \frac{2}{(s+1)(s+2)} \tag{2.7}$$

The roots of the denominator are distinct, since each factor is raised only to unity power. We can write the partial-fraction expansion as a sum of terms where each factor of the original denominator forms the denominator of each term, and constants, called *residues*, form the numerators. Hence,

$$F(s) = \frac{2}{(s+1)(s+2)} = \frac{K_1}{(s+1)} + \frac{K_2}{(s+2)} \tag{2.8}$$

To find K_1, we first multiply Eq. (2.8) by $(s+1)$, which isolates K_1. Thus,

$$\frac{2}{(s+2)} = K_1 + \frac{(s+1)K_2}{(s+2)} \tag{2.9}$$

Letting s approach -1 eliminates the last term and yields $K_1 = 2$. Similarly, K_2 can be found by multiplying Eq. (2.8) by $(s+2)$ and then letting s approach -2; hence, $K_2 = -2$.

Each component part of Eq. (2.8) is an $F(s)$ in Table 2.1. Hence, $f(t)$ is the sum of the inverse Laplace transform of each term, or

$$f(t) = (2e^{-t} - 2e^{-2t})u(t) \tag{2.10}$$

In general, then, given an $F(s)$ whose denominator has real and distinct roots, a partial-fraction expansion,

$$F(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{(s+p_1)(s+p_2)\cdots(s+p_m)\cdots(s+p_n)}$$
$$= \frac{K_1}{(s+p_1)} + \frac{K_2}{(s+p_2)} + \cdots + \frac{K_m}{(s+p_m)} + \cdots + \frac{K_n}{(s+p_n)} \tag{2.11}$$

can be made if the order of $N(s)$ is less than the order of $D(s)$. To evaluate each residue, K_i, we multiply Eq. (2.11) by the denominator of the corresponding partial fraction. Thus, if we want to find K_m, we multiply Eq. (2.11) by $(s+p_m)$ and get

$$(s+p_m)F(s) = \frac{(s+p_m)N(s)}{(s+p_1)(s+p_2)\cdots(s+p_m)\cdots(s+p_n)}$$
$$= (s+p_m)\frac{K_1}{(s+p_1)} + (s+p_m)\frac{K_2}{(s+p_2)} + \cdots + K_m + \cdots$$
$$+ (s+p_m)\frac{K_n}{(s+p_n)} \tag{2.12}$$

If we let s approach $-p_m$, all terms on the right-hand side of Eq. (2.12) go to zero except the term K_m, leaving

$$\frac{(s+p_m)N(s)}{(s+p_1)(s+p_2)\cdots(s+p_m)\cdots(s+p_n)}\bigg|_{s \to -p_m} = K_m \tag{2.13}$$

The following example demonstrates the use of the partial-fraction expansion to solve a differential equation. We will see that the Laplace transform reduces the task of finding the solution to simple algebra.

Example 2.3

Laplace Transform Solution of a Differential Equation

PROBLEM: Given the following differential equation, solve for $y(t)$ if all initial conditions are zero. Use the Laplace transform.

$$\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + 32y = 32u(t) \tag{2.14}$$

SOLUTION: Substitute the corresponding $F(s)$ for each term in Eq. (2.14), using Item 2 in Table 2.1, Items 7 and 8 in Table 2.2, and the initial conditions of $y(t)$ and $dy(t)/dt$ given by $y(0-) = 0$ and $\dot{y}(0-) = 0$, respectively. Hence, the Laplace transform of Eq. (2.14) is

$$s^2 Y(s) + 12sY(s) + 32Y(s) = \frac{32}{s} \tag{2.15}$$

Solving for the response, $Y(s)$, yields

$$Y(s) = \frac{32}{s(s^2 + 12s + 32)} = \frac{32}{s(s+4)(s+8)} \tag{2.16}$$

To solve for $y(t)$, we notice that Eq. (2.16) does not match any of the terms in Table 2.1. Thus, we form the partial-fraction expansion of the right-hand term and match each of the resulting terms with $F(s)$ in Table 2.1. Therefore,

$$Y(s) = \frac{32}{s(s+4)(s+8)} = \frac{K_1}{s} + \frac{K_2}{(s+4)} + \frac{K_3}{(s+8)} \tag{2.17}$$

where, from Eq. (2.13),

$$K_1 = \left.\frac{32}{(s+4)(s+8)}\right|_{s\to 0} = 1 \tag{2.18a}$$

$$K_2 = \left.\frac{32}{s(s+8)}\right|_{s\to -4} = -2 \tag{2.18b}$$

$$K_3 = \left.\frac{32}{s(s+4)}\right|_{s\to -8} = 1 \tag{2.18c}$$

Hence,

$$Y(s) = \frac{1}{s} - \frac{2}{(s+4)} + \frac{1}{(s+8)} \tag{2.19}$$

Since each of the three component parts of Eq. (2.19) is represented as an $F(s)$ in Table 2.1, $y(t)$ is the sum of the inverse Laplace transforms of each term. Hence,

$$y(t) = (1 - 2e^{-4t} + e^{-8t})u(t) \tag{2.20}$$

Students who are using MATLAB should now run ch2p1 through ch2p8 in Appendix B. This is your first MATLAB exercise. You will learn how to use MATLAB to (1) represent polynomials, (2) find roots of polynomials, (3) multiply polynomials, and (4) find partial-fraction expansions. Finally, Example 2.3 will be solved using MATLAB.

The $u(t)$ in Eq. (2.20) shows that the response is zero until $t = 0$. Unless otherwise specified, all inputs to systems in the text will not start until $t = 0$. Thus, output responses will also be zero until $t = 0$. For convenience, we will leave off the $u(t)$ notation from now on. Accordingly, we write the output response as

$$y(t) = 1 - 2e^{-4t} + e^{-8t} \tag{2.21}$$

TryIt 2.1

Use the following MATLAB and Control System Toolbox statement to form the linear, time-invariant (LTI) transfer function of Eq. (2.22).

F = zpk([ ], [−1 −2 −2], 2)

Case 2. Roots of the Denominator of *F(s)* Are Real and Repeated   An example of an $F(s)$ with real and repeated roots in the denominator is

$$F(s) = \frac{2}{(s+1)(s+2)^2} \tag{2.22}$$

The roots of $(s + 2)^2$ in the denominator are repeated, since the factor is raised to an integer power higher than 1. In this case, the denominator root at -2 is a *multiple root* of *multiplicity* 2.

We can write the partial-fraction expansion as a sum of terms, where each factor of the denominator forms the denominator of each term. In addition, each multiple root generates additional terms consisting of denominator factors of reduced multiplicity. For example, if

$$F(s) = \frac{2}{(s+1)(s+2)^2} = \frac{K_1}{(s+1)} + \frac{K_2}{(s+2)^2} + \frac{K_3}{(s+2)} \tag{2.23}$$

then $K_1 = 2$, which can be found as previously described. K_2 can be isolated by multiplying Eq. (2.23) by $(s + 2)^2$, yielding

$$\frac{2}{s+1} = (s+2)^2 \frac{K_1}{(s+1)} + K_2 + (s+2)K_3 \tag{2.24}$$

Letting s approach -2, $K_2 = -2$. To find K_3 we see that if we differentiate Eq. (2.24) with respect to s,

$$\frac{-2}{(s+1)^2} = \frac{(s+2)s}{(s+1)^2} K_1 + K_3 \tag{2.25}$$

TryIt 2.2

Use the following MATLAB statements to help you get Eq. (2.26).

numf = 2;
denf = poly([−1 −2 −2]);
[k,p,k] = residue...
 (numf,denf)

K_3 is isolated and can be found if we let s approach -2. Hence, $K_3 = -2$.

Each component part of Eq. (2.23) is an $F(s)$ in Table 2.1; hence, $f(t)$ is the sum of the inverse Laplace transform of each term, or

$$f(t) = 2e^{-t} - 2te^{-2t} - 2e^{-2t} \tag{2.26}$$

If the denominator root is of higher multiplicity than 2, successive differentiation would isolate each residue in the expansion of the multiple root.

In general, then, given an $F(s)$ whose denominator has real and repeated roots, a partial-fraction expansion,

$$F(s) = \frac{N(s)}{D(s)}$$

$$= \frac{N(s)}{(s+p_1)^r(s+p_2)\cdots(s+p_n)}$$

$$= \frac{K_1}{(s+p_1)^r} + \frac{K_2}{(s+p_1)^{r-1}} + \cdots + \frac{K_r}{(s+p_1)}$$

$$+ \frac{K_{r+1}}{(s+p_2)} + \cdots + \frac{K_n}{(s+p_n)} \tag{2.27}$$

can be made if the order of $N(s)$ is less than the order of $D(s)$ and the repeated roots are of multiplicity r at $-p_1$. To find K_1 through K_r for the roots of multiplicity greater than unity, first multiply Eq. (2.27) by $(s+p_1)^r$ getting $F_1(s)$, which is

$$F_1(s) = (s+p_1)^r F(s)$$

$$= \frac{(s+p_1)^r N(s)}{(s+p_1)^r(s+p_2)\cdots(s+p_n)}$$

$$= K_1 + (s+p_1)K_2 + (s+p_1)^2 K_3 + \cdots + (s+p_1)^{r-1}K_r$$

$$+ \frac{K_{r+1}(s+p_1)^r}{(s+p_2)} + \cdots + \frac{K_n(s+p_1)^r}{(s+p_n)} \tag{2.28}$$

Immediately, we can solve for K_1 if we let s approach $-p_1$. We can solve for K_2 if we differentiate Eq. (2.28) with respect to s and then let s approach $-p_1$. Subsequent differentiation will allow us to find K_3 through K_r. The general expression for K_1 through K_r for the multiple roots is

$$K_i = \frac{1}{(i-1)!} \frac{d^{i-1}F_1(s)}{ds^{i-1}}\bigg|_{s\to-p_1} \quad i = 1, 2, \ldots, r; \quad 0! = 1 \tag{2.29}$$

Case 3. Roots of the Denominator of *F(s)* Are Complex or Imaginary   An example of $F(s)$ with complex roots in the denominator is

$$F(s) = \frac{3}{s(s^2 + 2s + 5)} \tag{2.30}$$

This function can be expanded in the following form:

$$\frac{3}{s(s^2 + 2s + 5)} = \frac{K_1}{s} + \frac{K_2 s + K_3}{s^2 + 2s + 5} \tag{2.31}$$

K_1 is found in the usual way to be $\frac{3}{5}$. K_2 and K_3 can be found by first multiplying Eq. (2.31) by the lowest common denominator, $s(s^2 + 2s + 5)$, and clearing the fractions. After simplification with $K_1 = \frac{3}{5}$, we obtain

$$3 = \left(K_2 + \frac{3}{5}\right)s^2 + \left(K_3 + \frac{6}{5}\right)s + 3 \tag{2.32}$$

TryIt 2.3

Use the following MATLAB and Control System Toolbox statement to form the LTI transfer function of Eq. (2.30).

```
F=tf([3],[1 2 5 0])
```

Balancing coefficients, $\left(K_2 + \frac{3}{5}\right) = 0$ and $\left(K_3 + \frac{6}{5}\right) = 0$. Hence $K_2 = -\frac{3}{5}$ and $K_3 = -\frac{6}{5}$. Thus,

$$F(s) = \frac{3}{s(s^2 + 2s + 5)} = \frac{3/5}{s} - \frac{3}{5}\frac{s+2}{s^2 + 2s + 5} \tag{2.33}$$

The last term can be shown to be the sum of the Laplace transforms of an exponentially damped sine and cosine. Using Item 7 in Table 2.1 and Items 2 and 4 in Table 2.2, we get

$$\mathscr{L}[Ae^{-at}\cos \omega t] = \frac{A(s+a)}{(s+a)^2 + \omega^2} \tag{2.34}$$

Similarly,

$$\mathscr{L}[Be^{-at}\sin \omega t] = \frac{B\omega}{(s+a)^2 + \omega^2} \tag{2.35}$$

Adding Eqs. (2.34) and (2.35), we get

$$\mathscr{L}[Ae^{-at}\cos \omega t + Be^{-at}\sin \omega t] = \frac{A(s+a) + B\omega}{(s+a)^2 + \omega^2} \tag{2.36}$$

We now convert the last term of Eq. (2.33) to the form suggested by Eq. (2.36) by completing the squares in the denominator and adjusting terms in the numerator without changing its value. Hence,

$$F(s) = \frac{3/5}{s} - \frac{3}{5}\frac{(s+1) + (1/2)(2)}{(s+1)^2 + 2^2} \tag{2.37}$$

Comparing Eq. (2.37) to Table 2.1 and Eq. (2.36), we find

$$f(t) = \frac{3}{5} - \frac{3}{5}e^{-t}\left(\cos 2t + \frac{1}{2}\sin 2t\right) \tag{2.38}$$

In order to visualize the solution, an alternate form of $f(t)$, obtained by trigonometric identities, is preferable. Using the amplitudes of the cos and sin terms, we factor out $\sqrt{1^2 + (1/2)^2}$ from the term in parentheses and obtain

$$f(t) = \frac{3}{5} - \frac{3}{5}\sqrt{1^2 + (1/2)^2}e^{-t}\left(\frac{1}{\sqrt{1^2 + (1/2)^2}}\cos 2t + \frac{1/2}{\sqrt{1^2 + (1/2)^2}}\sin 2t\right) \tag{2.39}$$

Letting $1/\sqrt{1^2 + (1/2)^2} = \cos \phi$ and $(1/2)/\sqrt{1^2 + (1/2)^2} = \sin \phi$,

$$f(t) = \frac{3}{5} - \frac{3}{5}\sqrt{1^2 + (1/2)^2}e^{-t}(\cos \phi \cos 2t + \sin \phi \sin 2t) \tag{2.40}$$

or

$$f(t) = 0.6 - 0.671e^{-t}\cos(2t - \phi) \tag{2.41}$$

where $\phi = \arctan 0.5 = 26.57°$. Thus, $f(t)$ is a constant plus an exponentially damped sinusoid.

In general, then, given an $F(s)$ whose denominator has complex or purely imaginary roots, a partial-fraction expansion,

$$F(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{(s + p_1)(s^2 + as + b) \cdots}$$

$$= \frac{K_1}{(s + p_1)} + \frac{(K_2 s + K_3)}{(s^2 + as + b)} + \cdots \qquad (2.42)$$

can be made if the order of $N(s)$ is less than the order of $D(s)$ p_1 is real, and $(s^2 + as + b)$ has complex or purely imaginary roots. The complex or imaginary roots are expanded with $(K_2 s + K_3)$ terms in the numerator rather than just simply K_i, as in the case of real roots. The K_i's in Eq. (2.42) are found through balancing the coefficients of the equation after clearing fractions. After completing the squares on $(s^2 + as + b)$ and adjusting the numerator, $(K_2 s + K_3)/(s^2 + as + b)$ can be put into the form shown on the right-hand side of Eq. (2.36).

Finally, the case of purely imaginary roots arises if $a = 0$ in Eq. (2.42). The calculations are the same.

Another method that follows the technique used for the partial-fraction expansion of $F(s)$ with real roots in the denominator can be used for complex and imaginary roots. However, the residues of the complex and imaginary roots are themselves complex conjugates. Then, after taking the inverse Laplace transform, the resulting terms can be identified as

$$\frac{e^{j\theta} + e^{-j\theta}}{2} = \cos \theta \qquad (2.43)$$

and

$$\frac{e^{j\theta} - e^{-j\theta}}{2j} = \sin \theta \qquad (2.44)$$

For example, the previous $F(s)$ can also be expanded in partial fractions as

$$F(s) = \frac{3}{s(s^2 + 2s + 5)} = \frac{3}{s(s + 1 + j2)(s + 1 - j2)}$$

$$= \frac{K_1}{s} + \frac{K_2}{s + 1 + j2} + \frac{K_3}{s + 1 - j2} \qquad (2.45)$$

Finding K_2,

$$K_2 = \frac{3}{s(s + 1 - j2)}\bigg|_{s \to -1-j2} = -\frac{3}{20}(2 + j1) \qquad (2.46)$$

Similarly, K_3 is found to be the complex conjugate of K_2, and K_1 is found as previously described. Hence,

$$F(s) = \frac{3/5}{s} - \frac{3}{20}\left(\frac{2 + j1}{s + 1 + j2} + \frac{2 - j1}{s + 1 - j2}\right) \qquad (2.47)$$

from which

$$f(t) = \frac{3}{5} - \frac{3}{20}\left[(2 + j1)e^{-(1+j2)t} + (2 - j1)e^{-(1-j2)t}\right]$$

$$= \frac{3}{5} - \frac{3}{20}e^{-t}\left[4\left(\frac{e^{j2t} + e^{-j2t}}{2}\right) + 2\left(\frac{e^{j2t} + e^{-j2t}}{2j}\right)\right] \qquad (2.48)$$

TryIt 2.5

Use the following MATLAB statements to help you get Eq. (2.47).

```
numf = 3
denf = [1 2 5 0]
[k,p,k] = residue...
  (numf,denf)
```

Using Eqs. (2.43) and (2.44), we get

$$f(t) = \frac{3}{5} - \frac{3}{5}e^{-t}\left(\cos 2t + \frac{1}{2}\sin 2t\right) = 0.6 - 0.671e^{-t}\cos(2t - \phi) \tag{2.49}$$

where $\phi = \arctan 0.5 = 26.57°$.

Symbolic Math
SM

```
Students who are performing the MATLAB exercises and want to
explore the added capability of MATLAB's Symbolic Math Toolbox
should now run ch2sp1 and ch2sp2 in Appendix F at www.wiley.com/
college/nise. You will learn how to construct symbolic objects and
then find the inverse Laplace and Laplace transforms of frequency
and time functions, respectively. The examples in Case 2 and Case 3
in this section will be solved using the Symbolic Math Toolbox.
```

Skill-Assessment Exercise 2.1

PROBLEM: Find the Laplace transform of $f(t) = te^{-5t}$.

ANSWER: $F(s) = 1/(s+5)^2$

The complete solution is at www.wiley.com/college/nise.

Skill-Assessment Exercise 2.2

PROBLEM: Find the inverse Laplace transform of $F(s) = 10/[s(s+2)(s+3)^2]$.

WileyPLUS
WPCS
Control Solutions

ANSWER: $f(t) = \dfrac{5}{9} - 5e^{-2t} + \dfrac{10}{3}te^{-3t} + \dfrac{40}{9}e^{-3t}$

The complete solution is at www.wiley.com/college/nise.

2.3   The Transfer Function

In the previous section we defined the Laplace transform and its inverse. We presented the idea of the partial-fraction expansion and applied the concepts to the solution of differential equations. We are now ready to formulate the system representation shown in Figure 2.1 by establishing a viable definition for a function that algebraically relates a system's output to its input. This function will allow separation of the input, system, and output into three separate and distinct parts, unlike the differential equation. The function will also allow us to *algebraically* combine mathematical representations of subsystems to yield a total system representation.

Let us begin by writing a general nth-order, linear, time-invariant differential equation,

$$a_n\frac{d^n c(t)}{dt^n} + a_{n-1}\frac{d^{n-1}c(t)}{dt^{n-1}} + \cdots + a_0 c(t) = b_m\frac{d^m r(t)}{dt^m} + b_{m-1}\frac{d^{m-1}r(t)}{dt^{m-1}} + \cdots + b_0 r(t)$$

$$\tag{2.50}$$